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Axisymmetric convection between two rotating disks

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A real fluid is contained between two horizontal infinite disks which rotate about a common vertical axis with the same angular velocity. On the upper disk there is an axisymmetric non-uniform temperature distribution with a minimum at the point of intersection of the disk and the axis of rotation. The lower disk is insulated. It is assumed that inertial accelerations are negligible in comparison with Coriolis accelerations and that viscous effects are confined to Ekman layers at the disks. Outside the Ekman layers, therefore, since the motion is axisymmetric, the buoyancy forces, by the geostrophic approximation, drive only an azimuthal component of the velocity field which cannot alter the temperature field. Thus heat is convected only by the secondary circulation which is driven by the viscous forces of the Ekman layers. It is possible then for the secondary flow to be so small that heat is transferred by conduction processes.

This paper analyses the conditions necessary for either conduction or convection processes to predominate and the structure of the velocity and temperature fields in these different situations. In addition the separate effects of a temperature maximum on the upper disk and of replacing the upper disk by a stress-free surface are considered.

1. Introduction

One of the most familiar examples of natural convection occurs when a point source of heat is applied at the bottom of a closed vessel which contains fluid at rest, or when a local cold source is applied at the top. The circulation in a vertical plane takes place even if the density differences which are produced are very small, since no hydrostatic pressure distribution can balance the consequent horizontal variation of the buoyancy forces. Normally, heat is convected by the moving particles of fluid: heat conduction and viscous effects are significant only in regions close to the fluid boundaries.

But in a rotating fluid the effects of density variations in a horizontal plane can be markedly different. For example, to an observer in a system which rotates about a vertical axis it seems that fluid is driven horizontally both parallel and perpendicular to any horizontal pressure gradient. If the temperature differences are not too large, then the ratio of the inertia to the Coriolis accelerations, the Rossby number, is small and this perpendicular component of the motion predominates. Thus, in contrast to the situation described above, the primary effect of density variations in a horizontal plane is not to produce a vertical circulation but a horizonal flow perpendicular to the gradient of the density. In particular, if all quantities are symmetric about the axis of rotation, this geostrophic flow is the component of velocity round that axis and thus cannot alter the temperature field by convective processes. Heat is effectively convected, therefore, only by the circulation in an axial plane, and this flow may be inhibited to such an extent by the geostrophic velocity that conduction processes predominate everywhere.

Robinson (1959) discussed an example of this type of flow in a theoretical investigation of the rotating annulus experiments of Hide (1958) and Fultz *et al.* (1959). A real fluid was contained in a horizontal annulus of square cross-section which rotated about its vertical axis. The inner and outer sides of the annulus were held at different constant temperatures, and the top and bottom were insulated. He investigated the axisymmetric régime which was observed in the experiments, and which was believed to be geostrophic, and found that for a particular range of the governing parameters the temperature distribution was in fact determined predominantly by conduction. This temperature field produced a geostrophic flow round the annulus, and the circulation in an axial plane was confined to boundary layers close to all four sides.

In this paper axisymmetric convection in a rotating fluid is examined in its simplest form, namely when the fluid is contained between two infinite horizontal disks which rotate about a vertical axis with the same angular velocity. The heat source is supplied on the upper disk in the form of an axisymmetric temperature distribution which has a minimum at the axis of rotation, and the bottom disk is insulated. We shall investigate what conditions are required, in addition to a small Rossby number, for the two fundamentally different methods of heat transfer, conduction and convection, to predominate, and what the corresponding structures of the velocity and temperature fields are. Although not directly related to the experiments mentioned above, for the heat sources are applied on different boundaries, it will become clear that this situation does describe to a considerable extent the axisymmetric geostrophic régime which was observed.

Von Kármán (1921) introduced a similarity solution for the flow induced in a homogeneous semi-infinite fluid by an infinite rotating disk, and this was generalized by Batchelor (1951) for the flow between two parallel infinite disks which rotate about a common axis with different speeds. The remarkable property of the solution is that all the terms which are normally neglected in the boundary-layer approximations vanish identically from the Navier–Stokes equations. If the temperature distribution in our model is chosen to be of an appropriate form, then the similarity profiles of these velocity fields hold, and the problem reduces to the solution of a set of non-linear ordinary differential equations.

2. The equations of motion

Consider a viscous conducting fluid contained between two infinite horizontal disks which rotate in a constant gravitational field about a common vertical axis with angular velocity Ω. The disks are a distance d apart. An axially symmetric temperature distribution which increases radially from the axis of rotation is imposed on the upper disk, and the lower disk is thermally insulated. The point of intersection of the axis of rotation and the upper disk we shall call the pole.

Let a cylindrical polar co-ordinate system (r, θ, z) be fixed in the lower disk with the origin situated at the point of intersection of that disk and the axis of rotation, and with the z-axis pointing vertically upwards. The fluid thus lies between $z = 0$ and $z = d$.

Since the density changes by only a small amount from the density ρ_0 at the pole we assume the Boussinesq approximation: namely

$$\rho = \rho_0\{1-\alpha(T-T_0)\},$$

where T_0 and T are the temperatures at which the density is ρ_0 and ρ respectively, and the coefficient of thermal expansion α is taken to be a negligibly small constant except when multiplied by gravity. This means that in the momentum equations density variations appear only as variations in buoyancy forces. It also implies that the continuity equation reduces to

$$\partial u/\partial r + u/r + \partial w/\partial z = 0. \tag{2.1}$$

We rewrite the pressure P in the fluid as $P = p' + p^*$, where $\partial p^*/\partial z = -g\rho_0$. The quantity p' is thus the deviation from the hydrostatic pressure of a fluid with uniform density ρ_0. We assume that viscosity and thermal conductivity are constant. Then, with respect to the rotating co-ordinate frame of reference, the Navier–Stokes equations for steady laminar flow are

$$(\mathbf{u}.\nabla)\,\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = -\rho_0^{-1}\nabla p' + \nu\nabla^2\mathbf{u} + g\alpha(T-T_0)\,\mathbf{k}, \tag{2.2}$$

where $\mathbf{k}$ is a unit vector in the z-direction. Now the centrifugal forces can be expressed as a gradient:

$$\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = \nabla(-\tfrac{1}{2}\Omega^2 D^2),$$

where D is the distance of a particle of fluid from the axis of rotation. We define the reduced pressure p by

$$p = p'/\rho_0 - \tfrac{1}{2}\Omega^2 D^2,$$

and (2.2) becomes

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - 2\Omega v = -\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right), \tag{2.3}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + 2\Omega u = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right), \tag{2.4}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) + g\alpha(T-T_0). \tag{2.5}$$

This form of the equations is suitable provided that no boundary conditions are given on the absolute pressure P. The viscous dissipation is neglected in the energy equation

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right). \tag{2.6}$$

The boundary conditions of no slip and zero heat-flux at the lower disk are

$$u = v = w = \partial T/\partial z = 0 \quad \text{at} \quad z = 0. \tag{2.7}$$

We now use von Kármán's fundamental assumption that the vertical velocity is a function of z only. There will therefore be no closed cells in the flow in a vertical plane through the axis of rotation. Then the continuity equation (2.1) immediately gives

$$u = -\tfrac{1}{2} r\, dw/dz \tag{2.8}$$

if u is finite at the origin. Substituting this result in (2.5) and integrating we find

$$p = -\tfrac{1}{2}w^2 + \nu\frac{dw}{dz} + g\alpha \int_0^z (T-T_0)\, dz + \pi(r), \tag{2.9}$$

where $\pi(r)$ is determined by the conditions on the lower disk. Insert (2.9) in equation (2.3) to give

$$\left(\frac{1}{2}\frac{dw}{dz}\right)^2 - \tfrac{1}{2}w\frac{d^2w}{dz^2} - \left(\frac{v}{r}\right)^2 - 2\Omega\frac{v}{r} = -\frac{g\alpha}{r}\int_0^z \frac{\partial}{\partial r}(T-T_0)\, dz - \frac{1}{r}\frac{d}{dr}\pi(r) - \tfrac{1}{2}\nu\frac{d^3w}{dz^3}. \tag{2.10}$$

On $z = 0$, by (2.7) and (2.8), $v = w = dw/dz = 0$. Thus since the integral disappears

$$0 = -\frac{1}{r}\frac{d}{dr}\pi(r) + \text{const.}, \quad \text{i.e.} \quad \pi(r) = \tfrac{1}{2}c_1 r^2 + c_2,$$

where c_1 and c_2 are constants. Then, by (2.10),

$$-\left(\frac{v}{r}\right)^2 - 2\Omega\frac{v}{r} + \frac{g\alpha}{r}\int_0^z \frac{\partial}{\partial r}(T-T_0)\, dz = \text{function of } z \text{ only}.$$

Hence, for the temperature distribution to be consistent with von Kármán's similarity solution, that is, for v/r to be a function of z only, it is essential that

$$\frac{1}{r}\int_0^z \frac{\partial}{\partial r}(T-T_0)\, dz = \text{function of } z \text{ only},$$

$$\text{i.e.} \quad T - T_0 = \tfrac{1}{2}r^2 f(z) + h(z). \tag{2.11}$$

Equation (2.10) therefore becomes

$$\left(\frac{1}{2}\frac{dw}{dz}\right)^2 - \tfrac{1}{2}w\frac{d^2w}{dz^2} - \left(\frac{v}{r}\right)^2 - 2\Omega\left(\frac{v}{r}\right) = -g\alpha\int_0^z f(z)\, dz - c_1 - \tfrac{1}{2}\nu\frac{d^3w}{dz^3}. \tag{2.12}$$

By the similarity of the problem, if we differentiate this equation with respect to z we are in fact eliminating the pressure, and the resulting azimuthal component of the vorticity equation is given below in (2.13).

Equation (2.4) reduces to the form given by (2.14) and the energy equation (2.6) becomes

$$-\tfrac{1}{2}r^2\frac{dw}{dz}f + \tfrac{1}{2}wr^2\frac{df}{dz} + w\frac{dh}{dz} = \kappa\left(2f(z) + \tfrac{1}{2}r^2\frac{d^2f}{dz^2} + \frac{d^2h}{dz^2}\right).$$

This can happen only if the coefficients of r^2 and r^0 are separately satisfied. The two components are shown in (2.15) and (2.16).

The governing equations thus reduce to four non-linear differential equations:

$$-\tfrac{1}{2}w\frac{d^3w}{dz^3}-2\frac{v}{r}\frac{d}{dz}\left(\frac{v}{r}\right)-2\Omega\frac{d}{dz}\left(\frac{v}{r}\right)=-g\alpha f(z)-\frac{\nu}{2}\frac{d^4w}{dz^4}, \tag{2.13}$$

$$-\frac{dw}{dz}\frac{v}{r}+w\frac{d}{dz}\left(\frac{v}{r}\right)-\Omega\frac{dw}{dz}=\nu\frac{d^2}{dz^2}\left(\frac{v}{r}\right), \tag{2.14}$$

$$-\frac{dw}{dz}f+w\frac{df}{dz}=\kappa\frac{d^2f}{dz^2}, \tag{2.15}$$

$$w\frac{dh}{dz}=\kappa\left(2f+\frac{d^2h}{dz^2}\right). \tag{2.16}$$

From the equations (2.13) and (2.14) it is clear that the horizontal temperature gradient drives the motion, for $f(z)$, by (2.11), is a measure of this gradient. This component of the temperature, by the first and second terms respectively of (2.15), is convected radially and vertically; whereas the other component $h(z)$, by (2.16), is convected only vertically. The first three equations can be solved independently of the fourth and so, in this sense, $h(z)$ is determined by forced vertical convection. It follows then that an upward flow along the axis could occur even if the temperature at the pole was less than the temperature at the origin. This and other possible anomalies are simply a result of the infinite radii of the two disks. In setting up the boundary conditions on the temperature, therefore, care must be taken to create a physically sensible situation.

The boundary conditions at the lower disk $z = 0$ are, by (2.7) and (2.8),

$$v/r = w = dw/dz = 0, \tag{2.17}$$

and the zero heat-flux condition must be applied to both temperature components

$$df/dz = dh/dz = 0. \tag{2.18}$$

At the upper disk $z = d$ the temperature components f and h have fixed values since the temperature is known, and

$$v/r = w = dw/dz = 0. \tag{2.19}$$

3. Order-of-magnitude considerations

In this section we shall deduce, by order-of-magnitude considerations, the structure of the velocity and temperature fields for the different situations of conduction and convection domination. Then in § 4 the analytical solutions for conduction domination will be described, and in § 5 solutions will be postulated for the situation when convection processes become important.

3.1. *The governing dimensionless parameters*

The simplest procedure for analysing orders of magnitude is to reduce the equations to non-dimensional form, so that the orders of magnitude are expressed in terms of dimensionless parameters. The only length scale in the problem is the distance d between the two disks. For a temperature scale we choose the difference in temperature between that at the pole, T_0, and the temperature T_d

at a point on the upper disk distant $\sqrt{2d}$ from the pole: $\Delta T = T_d - T_0 > 0$. This particular temperature scale ensures that the dimensionless form of f takes the value unity on the upper disk. Although there is no natural velocity scale one can be formed from the geostrophic balance which we expect to occur between the buoyancy and Coriolis forces. Thus if c is this scale

$$c = g\alpha\,\Delta T/\Omega. \tag{3.1}$$

This velocity, then, represents the change in the horizontal geostrophic flow between two points which are a vertical distance d apart, when the horizontal temperature gradient is uniform and is $2\Delta T/d$.

Thus define a new vertical co-ordinate and new velocities and temperature functions

$$\zeta = z/d, \quad v/r = \overline{V}(\zeta)\,g\alpha\Delta T/(\Omega d), \quad w = \overline{W}(\zeta)\,g\alpha\Delta T/\Omega,$$
$$T - T_0 = \tfrac{1}{2}r^2 f + h = (\tfrac{1}{2}r^2 d^{-2}\overline{F} + \overline{H})\Delta T.$$

We shall call $\overline{V}(\zeta)$ the zonal velocity although it is in fact the angular velocity of any particle.

The equations (2.13)–(2.16) now become dependent on three dimensionless parameters:

$$\beta(2\overline{V}\,\overline{V}' + \tfrac{1}{2}\overline{W}\,\overline{W}''') + 2\overline{V}' = \overline{F} + \tfrac{1}{2}\epsilon\overline{W}^{\text{iv}}, \tag{3.2}$$

$$\beta(-\overline{W}'\overline{V} + \overline{W}\,\overline{V}') - \overline{W}' = \epsilon\overline{V}'', \tag{3.3}$$

$$\sigma\beta(-\overline{W}'\overline{F} + \overline{W}\,\overline{F}') = \epsilon\overline{F}'', \tag{3.4}$$

$$\sigma\beta\overline{W}\,\overline{H}' = \epsilon(2\overline{F} + \overline{H}''), \tag{3.5}$$

where primes denote derivatives with respect to ζ, and

$$\sigma = \nu/\kappa, \quad \beta = g\alpha\,\Delta T/(d\Omega^2), \quad \epsilon = \nu/(d^2\Omega).$$

The parameter σ is the Prandtl number, which is of order 10 for water and unity for air, and β is a Rossby number based on the velocity scale (3.1). We can see in these equations that the relative importance of viscous and conduction processes is represented by ϵ, which is the inverse square root of a Taylor number. From the discussion of § 1 it is clear that we are interested in the régime of flow represented by small β and ϵ. We shall not be concerned with the effects which variations in σ produce. Thus we shall assume that

$$\sigma = O(1), \quad \beta, \epsilon \ll 1.$$

The boundary conditions (2.17)–(2.19) in dimensionless form are

$$\text{at} \quad \zeta = 0, \quad \overline{V} = \overline{W} = \overline{W}' = \overline{F}' = \overline{H}' = 0; \tag{3.6}$$

$$\text{at} \quad \zeta = 1, \quad \overline{V} = \overline{W} = \overline{W}' = 0, \quad \overline{F} = 1, \quad \overline{H} = 0. \tag{3.7}$$

Since the parameters β and ϵ are small it is natural to investigate first the effects of setting them equal to zero, when equations (3.2) and (3.3) become

$$2\overline{V}' = \overline{F}, \quad \overline{W}' = 0. \tag{3.8}$$

The second of these, by the relation (2.8), shows that the flow in an axial plane is parallel to the axis of rotation, a result which is common to all inviscid axisymmetric flows at low Rossby number. In the first equation we have the familiar thermal wind equation which demonstrates that the horizontal temperature gradient drives only the zonal component of the velocity. Now, since the motion is axisymmetric, the zonal velocity plays no part in convecting the heat, as is clear from (3.4) and (3.5). Thus heat is convected entirely by the flow in an axial plane, and this motion must have as its driving mechanism the viscous or inertia terms which we have neglected in deriving (3.8). In spite of the fact then that we expect the conduction terms in (3.4) and (3.5) to be small, except in singular regions, it is possible for the velocities in an axial plane to be so low that heat is transported by conduction processes, as was pointed out in § 1.

3.2. *Conduction domination*

Let us assume first of all that conduction processes do predominate, so that the temperature field to a first approximation is determined by neglecting the left-hand sides of (3.4) and (3.5). The isotherms are then as shown in figure 1, and

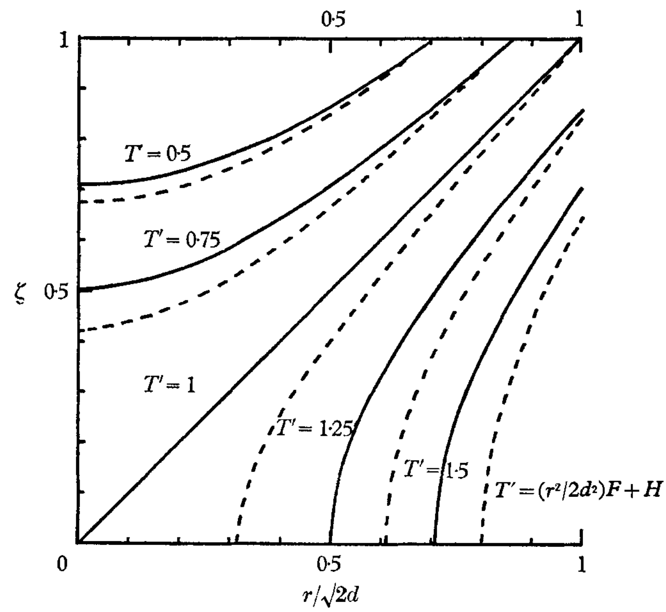

FIGURE 1. Isotherms for conduction domination. Broken curves are first-order isotherms for $\beta\sigma/4\sqrt{\epsilon} = 10^{-1}$, $\epsilon = 10^{-2}$.

we can expect that the equations (3.8) will hold throughout most of the region between the disks. Close to the disks, however, singular regions are essential to provide a driving mechanism for the secondary flow. In these regions the vertical gradients of velocity become large so that the Coriolis forces are balanced by viscous forces. Thus each region is the familiar Ekman layer which has a thickness of order $\sqrt{(\nu/\Omega)}$, or in dimensionless terms of order $\epsilon^{\frac{1}{2}}$.

It is well known that, if a real fluid flows at low Rossby number past a horizontal plane which rotates about a vertical axis, then there is a vertical velocity w

induced away from the plane given by

$$w = \tfrac{1}{2}\sqrt{(\nu/\Omega)}\,(\operatorname{curl}\mathbf{u}_h)_z,$$

where Cartesian axes with the z-axis vertical rotate with the plane, and where $\mathbf{u}_h(x, y)$ is the velocity of the fluid far from the plane. In our cylindrical polar co-ordinate system this gives

$$w = \frac{1}{2}\sqrt{\left(\frac{\nu}{\Omega}\right)}\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right),$$

or in dimensionless terms, since the radial velocity by (3.8) vanishes outside any viscous layers,

$$\overline{W} = \sqrt{\epsilon}\,\overline{V}. \tag{3.9}$$

This relationship gives the vertical velocity away from both the boundary layers as can be verified in the solutions of §4.1.

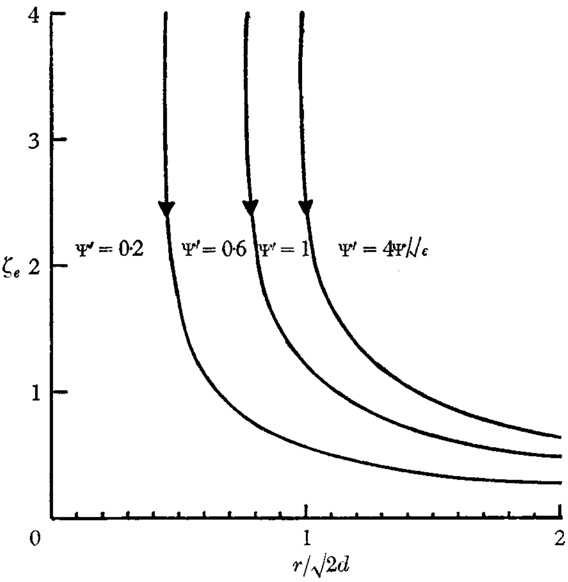

FIGURE 2. Zero-order streamlines of secondary flow at lower disk.

The flow field can therefore be divided into three regions, namely an inviscid core bounded by Ekman layers at each disk. In the core the horizontal temperature gradient $\overline{F}$, by (3.8), drives a unit-order zonal velocity, the change in which in a distance of order $\epsilon^{\frac{1}{2}}$ is very small. Hence by the Ekman layer suction condition (3.9) the vertical velocity $\overline{W}$ is of order $\epsilon^{\frac{1}{2}}$ and is proportional to the value which the thermal wind would have at the boundaries. But $\overline{W}$ is constant in any inviscid region of the flow. Thus, if the flow out of one Ekman layer is to match the flow into the other, these two values of $\overline{V}$ must be equal in magnitude and opposite in sign. Also, since, from figure 1, the horizontal temperature gradient is positive, the vertical gradient of $\overline{V}$ is positive. Consequently the condition (3.9) gives a down-draught of fluid. The streamlines of the secondary flow are shown in figure 2.

That the singular layers are in fact proper Ekman layers to a first approximation is now clear, for on substituting the orders of magnitude of $\overline{V}$ and $\overline{W}$ in equations (3.2) and (3.3), and remembering that the relevant length scale is of order $\epsilon^{\frac{1}{2}}$, we have the following respective orders of magnitude of the various terms:

$$\beta\epsilon^{-\frac{1}{2}} \qquad \beta\epsilon^{-\frac{1}{2}} \qquad \epsilon^{-\frac{1}{2}} \qquad 1 \qquad \epsilon^{-\frac{1}{2}},$$
$$\beta \qquad \beta \qquad 1 \qquad 1.$$

Since β is small it is obvious that the inertia and buoyancy terms are negligible in the Ekman layers. What is significant here is that the relative importance of the inertia terms is determined solely by the size of the thermal Rossby number β. Provided that it is small, and that $\overline{V}$ and $\overline{W}$ are of respective orders 1 and $\epsilon^{\frac{1}{2}}$, the inertia terms are negligible to a first approximation.

In the heat-transfer equation (3.4), however, the situation is entirely different. (We can neglect (3.5) for the moment since (3.2)–(3.4) form a closed set of equations.) In the Ekman layers, order-of-magnitude considerations show that the terms in (3.4) have size

$$\sigma\beta \qquad \sigma\beta \qquad 1 \tag{3.10}$$

so that conduction dominates. In the core, however, the sizes of the terms are

$$\sigma\beta\epsilon^{\frac{1}{2}} \qquad \sigma\beta\epsilon^{\frac{1}{2}} \qquad \epsilon, \tag{3.11}$$

so that conduction dominates only if

$$\sigma\beta\epsilon^{-\frac{1}{2}} \ll 1. \tag{3.12}$$

This then is an extra constraint on the parameters for our assumption of conduction domination to be valid. But, since the inertia terms are always small when β is small, if we reverse the condition (3.12) and set $\sigma\beta\epsilon^{-\frac{1}{2}} \gg 1$, then it seems that the structure of the flow field will not change. The only difference will be that heat is convected in the core by a vertical velocity of order $\epsilon^{\frac{1}{2}}$. This example will be investigated in the next section.

3.3. *Convection domination*

The final remarks of the last section indicate that Ekman layer suction is the sole driving mechanism of the secondary flow when

$$\sigma\beta\epsilon^{-\frac{1}{2}} \gg 1. \tag{3.13}$$

Let this condition hold and let us assume that, by some means as yet unknown, a temperature field exists in the fluid and drives a unit-order zonal velocity according to (3.8). This thermal wind then causes a vertical velocity of order $\epsilon^{\frac{1}{2}}$ by the Ekman layer suction condition (3.9), and by (3.13) convection dominates in the core. Also, the orders of magnitude of the terms of (3.4) are still given by (3.10) and hence conduction dominates in the Ekman layers.

There must therefore be an intermediate thermal layer of thickness δ_T, say, where conduction and convection balance. The orders of magnitude of (3.4) in this thermal layer are thus

$$\sigma\beta\epsilon^{\frac{1}{2}}/\delta_T \qquad \sigma\beta\epsilon^{\frac{1}{2}}/\delta_T \qquad \epsilon/\delta_T^2$$

and a balance occurs if $\delta_T = \sigma^{-1}\beta^{-1}\epsilon^{\frac{1}{2}}$, which, by (3.13), represents a very thin layer, yet one which is thicker than an Ekman layer. The real meaning of the parameter $\sigma\beta\epsilon^{-\frac{1}{2}}$ in (3.12) and (3.13) is now apparent, for the condition $\sigma\beta\epsilon^{-\frac{1}{2}} \ll 1$ implies that the thermal layer is very much thicker than the distance between the disks, and thus conduction dominates everywhere. On the other hand, if $\epsilon^{\frac{1}{2}}\beta^{-1}\sigma^{-1} = \delta_T \ll 1$, then a very thin thermal layer exists with consequent transfer of heat by convection in the core.

The velocity and temperature fields can thus be divided into five regions, namely an inviscid convecting core bounded by thermal layers which in turn are bounded by Ekman layers. In the core the vertical velocity is constant and thus by (3.4) and (3.5) the temperature components $\overline{F}$ and $\overline{H}$ are constant. The isotherms are therefore vertical, and consequently the zonal velocity is a linear function of the vertical co-ordinate ζ.

Since conduction dominates in the Ekman layers, it might seem that $\overline{F}$ and $\overline{H}$ are linear functions of the appropriate boundary-layer co-ordinates $\zeta/\sqrt{\epsilon}$ and $(\zeta-1)/\sqrt{\epsilon}$. Such functions, however, are not bounded at the internal edges of the Ekman layers and must be rejected. Thus $\overline{F}$ and $\overline{H}$ are constant there.

In the thermal layers, order-of-magnitude considerations show that the Coriolis term in (3.2) is the largest, so that, to a first approximation, the zonal velocity is constant. Our assumption that the vertical velocity is constant both in the core and thermal layers is confirmed in a similar manner. Thus equation (3.4) is

$$\sigma\beta\overline{W}\overline{F}' = \epsilon\overline{F}'',$$

which integrates to give

$$\overline{F} = A\exp(\sigma\beta\overline{W}\zeta/\epsilon)+B \quad \text{and} \quad \overline{F} = C\exp(\sigma\beta\overline{W}(\zeta-1)/\epsilon)+D$$

in the lower and upper thermal layers, respectively, where A, B, C, and D are constants. But, since $\overline{W}$ is constant outside the Ekman layers, only one of these solutions can be bounded as ζ approaches values in the core, i.e. as ζ/δ_T becomes large and positive and $(\zeta-1)/\delta_T$ becomes large and negative. Clearly the bounded temperature belongs to the layer into which the vertical velocity flows from the core and thus the other layer cannot exist, at least to this order. Which layer does exist therefore depends on the sign of $\overline{W}$ and this in turn is determined by the Ekman layer suction condition (3.9), the appropriate values of $\overline{V}$ being determined from the thermal wind equation (3.8). It seems reasonable to expect that $\overline{F}$ will again be positive, which immediately implies a positive vertical gradient of the zonal velocity $\overline{V}$. Again the values which the thermal wind would have at the boundaries must be equal in magnitude and opposite in sign, and thus there must be a down-draught. Consequently there is no upper thermal layer to this order, and the net result is that the temperature field on the upper disk is swept down by the secondary flow to the upper edge of the lower thermal layer.

Since the lower disk is insulated, the function of the lower thermal layer is to ensure that the vertical flux of heat in the core into a control annulus balances the outward radial flux of heat in the Ekman layer. This is shown schematically in figure 3. (I am grateful to Dr F. P. Bretherton for this interpretation.) Thus in dimensional terms, if the quantities in the core and in the Ekman layer are denoted respectively by the absence of subscripts, and by subscripted e's, we have

$$-w\left(\frac{r^2}{2d^2}\bar{F}+\bar{H}\right)\Delta T\,2\pi r\,\delta r=\left[\int_0^{\delta_e}u_e\left(\frac{r^2}{2d^2}\bar{F}_e+\bar{H}_e\right)\Delta T\,2\pi r\,dz\right]_r^{r+\delta r}$$

$$=\left[\int_0^{\delta_e}-\frac{r^2}{2}\frac{dw_e}{dz}\left(\frac{r^2}{2d^2}\bar{F}_e+\bar{H}_e\right)\Delta T\,2\pi\,dz\right]_r^{r+\delta r}\quad\text{by (2.8)},$$

where δ_e is the thickness of the Ekman layer. The integral can be evaluated immediately, since we have shown that the temperature components are constant in the Ekman layers. Hence, since $\bar{F} = 1$, and $\bar{H} = 0$, it follows that

$$\bar{F}_e = \tfrac{1}{2},\quad \bar{H}_e = 0. \tag{3.14}$$

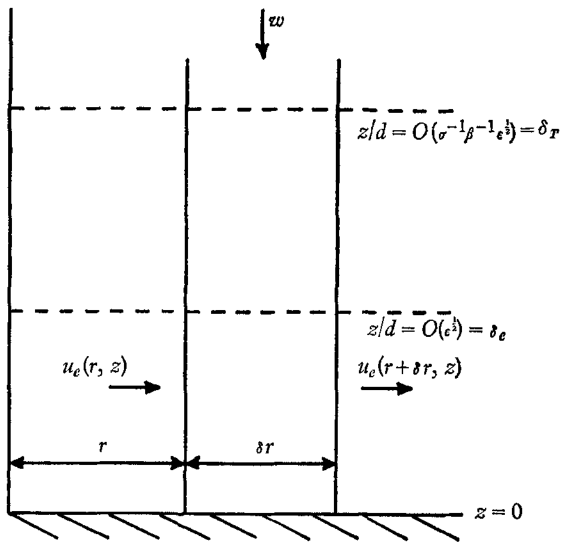

FIGURE 3. Conservation of heat, shown schematically.

3.4. *A temperature maximum at the pole*

In the last two sections it has become clear that the down-draught of fluid is caused by the combined effects of a temperature minimum at the pole, the thermal wind equation and Ekman layer suction. It seems reasonable to suppose that, if the temperature distribution on the upper disk has a maximum at the pole, then an up-draught of fluid will occur. Such a temperature maximum, however, means that the temperature scale of §3.1 must be altered to

$$\Delta T = T_0 - T_d > 0. \tag{3.15}$$

Thus we have
$$T - T_0 = (\tfrac{1}{2}r^2d^{-2}\bar{F}+\bar{H})\,\Delta T,$$

and the boundary conditions on the temperature at $\zeta = 1$ are

$$\bar{H} = 0,\quad \bar{F} = -1. \tag{3.16}$$

If conduction processes dominate then the temperature field to a first approximation is known everywhere. The structure of the flow field is the same as in

§3·2: the zonal velocity now has a negative vertical gradient in the core so that the Ekman layer suction condition (3.9) gives an up-draught of fluid. Clearly the extra constraint (3.12) must be imposed.

If convection processes dominate we must alter the temperature condition on the lower disk since the problem is no longer properly posed. For, if there is an up-draught of fluid from an insulated lower disk the temperature of the rising particles is not specified anywhere, a consequence of the infinite radii of the disks. Consistency can be achieved if we stipulate that the lower disk is maintained at a uniform temperature, i.e.

$$\text{at} \quad \zeta = 0, \quad \overline{F} = 0, \quad \overline{H} = h_0, \tag{3.17}$$

where h_0 is a negative constant. Even now the boundary conditions seem to be absurd, because we can always find a point on the top disk at which the temperature is less than h_0. This inconsistency also arises because of the nature of the similarity solution, and can be ignored by restricting our considerations to a suitable region close to the axis.

We might expect that the structure of the velocity and temperature fields will be identical with those of §3.3, but this is not so. For an up-draught of fluid implies that there can only be an upper thermal layer with the result that below this layer the fluid is homogeneous. Consequently by (3.8), since $\overline{F} = 0$, the zonal velocity is constant between the Ekman layers. But we have already discovered that to maintain a constant vertical velocity which links the Ekman layers the zonal velocity must change sign in the inviscid regions.

The anomaly arises because we have expected the velocities to be of the same order of magnitude as those of §§3.2 and 3.3. By carrying homogeneous fluid upwards, however, the secondary circulation is working against the downward diffusion of the non-uniform temperature field from the upper disk into the fluid. In other words, the convection tends to restore the system to equilibrium. We must therefore expect that there is a balance between the two opposing methods of heat transfer, and this balance results in smaller velocities than we had anticipated.

The correct orders of magnitude of $\overline{V}$ and $\overline{W}$ can be ascertained from the fact that the zonal velocity must change sign outside the Ekman layers. If we assume that the relative structure of the Ekman and thermal layers is maintained, the only possible way in which $\overline{V}$ can vary in the inviscid regions is by a geostrophic balance in the upper thermal layer. Thus, by equation (3.2), $\overline{V} = O(\delta'_T)$, where δ'_T is the thickness of this layer. Now if $\overline{W}$ is constant outside the Ekman layers, which are still of thickness $\delta_e = O(\epsilon^{\frac{1}{2}})$, we know from the Ekman layer suction condition (3.9) that $\overline{W} = O(\epsilon^{\frac{1}{2}}\overline{V}) = O(\epsilon^{\frac{1}{2}}\delta'_T)$. Hence the orders of magnitude of the terms of (3.4) in the thermal layer are

$$\sigma\beta\epsilon^{\frac{1}{2}}\delta'_T/\delta'_T \qquad \sigma\beta\epsilon^{\frac{1}{2}}\delta'_T/\delta'_T \qquad \epsilon/\delta'^2_T.$$

Since convection and conduction must balance we have

$$\delta'_T = \epsilon^{\frac{1}{4}}\beta^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}. \tag{3.18}$$

Thus
$$\overline{V} = O(\epsilon^{\frac{1}{4}}\beta^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}), \quad \overline{W} = O(\epsilon^{\frac{3}{4}}\beta^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}). \tag{3.19}$$

These orders of magnitude of the velocities now give a consistent solution.

3.5. *General considerations*

It is clear from the above discussion that the general structure of the velocity and temperature fields is not a direct consequence of the similarity solution. The result of von Kármán's assumptions is simply that the terms which would be neglected in the first approximation in fact vanish identically. Thus the order-of-magnitude considerations apply to any general axisymmetric flow between horizontal planes which rotate about a vertical axis, provided that the effects of vertical boundaries are negligible, and that the applied temperature gradient does not change sign. This latter constraint must apply, for it seems that singular layers parallel to the axis of rotation might develop if it is violated. Consider, for example, the axisymmetric flow between two horizontal rotating disks, at the upper one of which a temperature distribution is imposed with a gradient which oscillates about zero with distance from the axis of rotation. The lower disk is insulated and conditions are such that convection dominates. We might then expect there to be alternate regions of up-draught and down-draught. But, as we have seen, the velocities in the down-draught regions are an order of magnitude larger than the corresponding velocities for regions of up-draught. Consequently any upward motions are not 'sucked' by the surface temperature distribution but are forced by the downward flow. In regions of down-draught there is a lower thermal layer above which the isotherms are vertical so that the zonal velocity, by the thermal wind equation, varies linearly with depth. Fluid passes from the regions of down-draught to those of up-draught in the lower Ekman layer, where the temperature is constant with depth to a first approximation. Consider the simplest situation when the streams of fluid which enter a region of upward motion from opposite sides have the same temperature. Then in those regions there is an upper thermal layer below which the fluid is homogeneous with the result that the zonal velocity is constant with depth. Hence, where the regions of up-draught and down-draught meet, there must be vertical thermal layers which link the upper and lower horizontal thermal layers and which provide the necessary discontinuity in the temperature distribution. Also there must be vertical viscous layers to provide the necessary discontinuity in the zonal velocity. Of course, until a detailed analysis of such layers is made, the above argument is in doubt.

With regard to the rotating annulus experiments mentioned in § 1, it is obvious that the above deductions cannot be applied directly to the axisymmetric geostrophic régime, because in the experiments the temperature distributions were imposed on vertical boundaries. It is generally recognized that viscous and thermal layers occur at these vertical surfaces, but the manner in which they influence the flow in the core is not properly understood. From the above considerations, however, it seems reasonable to expect that it is the Ekman layers at the horizontal boundaries which control the flow and hence the temperature field in the core, and that the vertical layers which are parallel to the rotation axis play the passive role of channelling fluid from one Ekman layer to the other.

3.6. *A free top surface with a temperature minimum*

Lastly, it is interesting to retain the temperature boundary conditions of §3.1, and replace the upper disk by a free surface. This we define to be a surface on which the tangential viscous stresses vanish. Thus we have

$$\text{at} \quad z = d, \quad w = \partial u/\partial z = \partial v/\partial z = 0, \quad f \text{ and } h \text{ given,}$$

$$\text{i.e. at} \quad \zeta = 1, \quad \overline{W} = \overline{W}'' = \overline{V}' = 0, \quad \overline{F} = +1 \quad \overline{H} = 0. \tag{3.20}$$

Such a definition of course is only an approximation, since the position of a true free surface is not fixed but is a function of the motion of the fluid. But the definition is reasonable provided that the centrifugal forces and the slope of the true free surface are small. Once the solutions are obtained, the approximate position of the true free surface can be calculated in the following manner. Let the displacement of the free surface from the fixed free surface $z = d$ be $\xi(r)$ and let the solution for the zonal velocity at the fixed surface be $v(r)$. Then, since the centrifugal forces are negligible, $v(r)$ must be driven by the hydrostatic pressure gradient caused by the slope of the free surface,

$$2\Omega v(r) = -(1/\rho_0)\,\partial p^*/\partial r = g\,d\xi/dr. \tag{3.21}$$

The approximation is valid provided that $2\Omega v(r)/g$ is small.

If the inertia terms in (3.3) are neglected and the resulting equation integrated then

$$-\overline{W}(1) + \overline{W}(0) = \epsilon\{\overline{V}'(1) - \overline{V}'(0)\}.$$

Since all three terms $\overline{W}(1)$, $\overline{W}(0)$, and $\overline{V}'(1)$ vanish, there is no tangential zonal stress on the lower disk. If conduction dominates, it follows from the thermal wind equation (3.8) that the temperature field drives a zonal velocity which has a non-zero vertical derivative at the lower disk and the free surface. In other words, the thermal wind exerts a unit-order stress on these two surfaces. We can therefore divide the flow field into two parts: the zonal velocity driven by the thermal wind equation; and the zonal velocity and secondary flow which are caused by the application of a stress at the two boundaries of the fluid. This stress must be both equal in magnitude and opposite in sign to that produced by the thermal wind in order to satisfy the vanishing stress conditions. Now a result of Ekman layer theory is that the vertical velocity, which is induced away from a boundary by a wind stress **S** on the surface of an ocean, is given by

$$w = \tfrac{1}{2}(\nu/\Omega)\,\mathrm{curl}\,\mathbf{S}.$$

Since the stress exerted by the thermal wind is of unit order we can see that in dimensionless terms we have $\overline{W} = O(\epsilon)$. Inserting this in the heat transfer equation (3.4), the orders of magnitude for the core are

$$\sigma\beta\epsilon \qquad \sigma\beta\epsilon \qquad \epsilon$$

and thus conduction dominates provided only that β is small. When the surface is free, therefore, we cannot obtain convection domination with negligible inertia terms.

4. The solutions for conduction domination

4.1. *Flow between two rotating disks*

From the equations (3.2)–(3.5) it is clear that, if all the dependent variables are expressed as power series in β, the series inserted in the equations, and coefficients of the powers of β equated, then both the inertia and convection terms are omitted to the first order of approximation and we have conduction domination. This expansion can be carried out, however, only if we assume that $\epsilon = O(1)$ for small β. Since ϵ itself is small the validity of the expansion procedure is likely to be determined by a constraint which involves a combination of β and ϵ, as we discovered in §3.2. Put

$$\overline{V} = \sum_0^\infty \beta^n V_n, \quad \overline{W} = \sum_0^\infty \beta^n W_n, \quad \overline{F} = \sum_0^\infty \beta^n F_n, \quad \overline{H} = \sum_0^\infty \beta^n H_n. \tag{4.1}$$

When the expansions (4.1) are inserted in the two parts of the energy equation, the coefficients of β^0 give the zero-order temperature field:

$$F_0'' = 0, \quad 2F_0 + H_0'' = 0,$$

where the boundary conditions, by (3.6) and (3.7), are

$$F_0'(0) = H_0'(0) = 0, \quad F_0(1) = 1, \quad H_0(1) = 0.$$

The solutions are simply

$$F_0 = 1, \quad H_0 = 1 - \zeta^2, \tag{4.2}$$

and the temperature field, therefore, has a minimum value at the pole (see figure 1).

The coefficients of β^0 in (3.2) and (3.3), by (4.2), are

$$2V_0' = 1 + \tfrac{1}{2}\epsilon W_0^{\text{iv}}, \tag{4.3}$$

$$-W_0' = \epsilon V_0'', \tag{4.4}$$

with the boundary conditions that

$$\text{at} \quad \zeta = 0 \quad \text{and} \quad \zeta = 1, \quad W_0 = W_0' = V_0 = 0.$$

Thus the zero-order velocities are driven by the known zero-order temperature field. These velocities will, in turn, determine the first-order temperature field and this procedure will be repeated to all orders.

Although the equations (4.3) and (4.4) can be solved exactly with the appropriate conditions, it is more instructive to treat the problem as a boundary-layer one because we expect that viscous effects will be small in the body of the fluid. It is convenient to introduce boundary-layer co-ordinates $\zeta_e = \zeta/\sqrt{\epsilon}$ for the lower Ekman layer, and $\zeta_E = (\zeta - 1)/\sqrt{\epsilon}$ for the upper. The approximate solutions are then

$$V_0 = \tfrac{1}{4}(2\zeta - 1 + \exp(-\zeta_e)\cos\zeta_e - \exp\zeta_E \cos\zeta_E), \tag{4.5}$$

$$W_0 = -\tfrac{1}{4}\sqrt{\epsilon}\{1 - \exp(-\zeta_e)(\cos\zeta_e + \sin\zeta_e) - \exp\zeta_E(\cos\zeta_E - \sin\zeta_E)\}, \tag{4.6}$$

with errors in V_0 and W_0 of order $\exp(-1/\sqrt{\epsilon})$ and $\sqrt{\epsilon}\exp(-1/\sqrt{\epsilon})$, respectively.

Now from the continuity equation (2.1) and by (2.8) it follows that a dimensional stream function ψ can be defined by

$$\psi = -\tfrac{1}{2}r^2 w(z).$$

In dimensionless terms this becomes

$$\Psi = -\tfrac{1}{2}(r/d)^2\, \overline{W}(\zeta),$$

and thus the first term of (4.6) confirms the order-of-magnitude analysis of §3.2 that in the body of the fluid the flow in an axial plane is parallel to the rotation axis. Radial flow is confined to the boundary layers which are linked by the constant downward continuity current. Some of the zero-order streamlines are shown in figure 2.

From the equations (3.4) and (3.5) the coefficients of β give the equations for the first-order temperature field

$$\sigma(-W_0' F_0 + W_0 F_0') = \epsilon F_1'', \tag{4.7}$$

$$\sigma W_0 H_0' = \epsilon(2F_1 + H_1''). \tag{4.8}$$

The boundary conditions are,

$$\text{at} \quad \zeta = 0, \quad F_1' = H_1' = 0 \quad \text{and, at} \quad \zeta = 1, \quad F_1 = H_1 = 0,$$

and the approximate solutions are

$$F_1 = \tfrac{1}{4}(\sigma/\sqrt{\epsilon})(\zeta - 1) + \tfrac{1}{4}\sigma\{1 + \exp(-\zeta_e)\cos\zeta_e - \exp\zeta_E \cos\zeta_E\}, \tag{4.9}$$

$$\begin{aligned} H_1 = {} & \tfrac{1}{4}(\sigma/\sqrt{\epsilon})(\zeta^2 - 1) - \tfrac{1}{4}\sigma(\zeta^2 - 1) - \tfrac{1}{2}\sigma\sqrt{\epsilon}(\zeta - \tfrac{3}{2}) \\ & + \tfrac{1}{4}\sigma\epsilon \exp(-\zeta_e)\{\zeta_e(\sin\zeta_e - \cos\zeta_e) + 3\sin\zeta_e\} \\ & - \tfrac{1}{4}\sigma\sqrt{\epsilon}\exp\zeta_E(\cos\zeta_E + \sin\zeta_E) \\ & - \tfrac{1}{4}\sigma\epsilon\exp\zeta_E\{\zeta_E(\cos\zeta_E + \sin\zeta_E) - 3\sin\zeta_E\}, \end{aligned} \tag{4.10}$$

with errors of order $\exp(-1/\sqrt{\epsilon})$ and $\sqrt{\epsilon}\exp(-1/\sqrt{\epsilon})$, respectively. Note that the boundary-layer contributions to F_1 are much larger than the contributions to H_1. This is because the zero-order temperature components are convected in different directions. Since F_0 is a constant, it is convected radially, and this occurs only in the Ekman layers where the radial velocity is of order unity. On the other hand H_0 is convected vertically and the vertical velocity is everywhere of order $\sqrt{\epsilon}$. Some isotherms are shown in figure 1 in which it is clear that the secondary flow tends to push the zero-order isotherms down to the lower disk.

The equations for the first-order velocities are

$$2V_1' + 2V_0 V_0' + \tfrac{1}{2}W_0 W_0''' = F_1 + \tfrac{1}{2}\epsilon W_1^{\text{iv}} \tag{4.11}$$

and

$$-W_1' - W_0' V_0 + W_0 V_0' = \epsilon V_1'', \tag{4.12}$$

with the boundary conditions

$$\zeta = 0 \quad \text{and} \quad \zeta = 1, \quad V_1 = W_1 = W_1' = 0.$$

These two equations can be solved with errors of order $\exp(-1/\sqrt{\epsilon})$ simply by inserting the known solutions and integrating. But the primary driving mechanism of V_1 and W_1 can be understood more clearly if we examine, in turn, the terms in ascending orders of ϵ. For this reason divide the flow field into the three regions of §3.2: the lower Ekman layer, where the relevant co-ordinate is ζ_e and where variables are denoted by a subscripted e, e.g. $V_{0,e}$; the upper Ekman layer with co-ordinate ζ_E, where variables are denoted by a subscripted E, e.g. $V_{0,E}$; and the core of the fluid.

The largest terms in (4.11) must be of order $\epsilon^{-\frac{1}{2}}$, by (4.9)

$$2V_1' = \tfrac{1}{4}(\sigma/\sqrt{\epsilon})(\zeta-1).$$

Thus the largest component of V_1 is driven solely by the first-order horizontal temperature gradient. Since V_1 is of order $\epsilon^{-\frac{1}{2}}$ we expect that a form of Ekman layer suction will produce a vertical velocity W_1 of order unity. Both top and bottom boundary layers have then the form

$$2\sqrt{\epsilon}\,V_{1,e}' = \tfrac{1}{2}W_{1,e}^{\mathrm{iv}}, \quad -W_{1,e}' = \sqrt{\epsilon}\,V_{1,e}'',$$

where primes denote derivatives with respect to ζ_e. The boundary layers are then true Ekman layers for the largest terms of V_1 and W_1. The solutions are

$$\left.\begin{aligned} V_{1,E} &= -(\sigma/32\sqrt{\epsilon})(1-\exp\zeta_E\cos\zeta_E),\\ V_1 &= (\sigma/8\sqrt{\epsilon})(\tfrac{1}{2}\zeta^2-\zeta+\tfrac{1}{4}),\\ V_{1,e} &= (\sigma/32\sqrt{\epsilon})(1-\exp(-\zeta_e)\cos\zeta_e),\end{aligned}\right\}$$

$$\left.\begin{aligned} W_{1,E} &= (\sigma/32)\{1-\exp\zeta_E(\cos\zeta_E-\sin\zeta_E)\},\\ W_1 &= \sigma/32,\\ W_{1,e} &= (\sigma/32)\{1-\exp(-\zeta_e)(\cos\zeta_e+\sin\zeta_e)\}.\end{aligned}\right\}$$

The full non-linear terms enter into the boundary-layer equations for the next-order components of V_1 and W_1, which are of magnitude 1 and $\sqrt{\epsilon}$, respectively.

It is now a straightforward matter to examine the constraint on β and ϵ for the expansion procedure to be valid. Since all the terms in the solutions, apart from factors of ϵ, are of order unity, or less, the zero and first-order terms are a good approximation to the exact solution if

$$\beta\epsilon^{-\frac{1}{2}} \ll 1.$$

We have assumed, however, that $\sigma = O(1)$. The most general form of the constraint is thus

$$\sigma\beta\epsilon^{-\frac{1}{2}} \ll 1. \tag{4.13}$$

We note, therefore, that an alternative method of solution would be in the form of a simultaneous expansion of the variables in powers of β and $\sigma\beta\epsilon^{-\frac{1}{2}}$.

4.2. *Flow with a free top surface*

In §3.6 it was suggested that conduction processes would always dominate for small Rossby number if the upper disk were replaced by a free surface. Since the method of solution is exactly the same as the above, it seems appropriate to treat this example now. It is clear from the power series expansions in β that the zero-order temperature field will be that given by (4.2) and that the zero-order velocities will satisfy (4.3) and (4.4) with the boundary conditions, by (3.6) and (3.20),

$$\text{at} \quad \zeta = 0, \quad W_0 = W_0' = V_0 = 0,$$
$$\text{at} \quad \zeta = 1, \quad W_0 = W_0'' = V_0' = 0.$$

The approximate solutions are then

$$V_0 = \tfrac{1}{2}\{\zeta - \sqrt{\epsilon} + \sqrt{\epsilon}\exp(-\zeta_e)\cos\zeta_e - \tfrac{1}{2}\sqrt{\epsilon}\exp\zeta_E(\cos\zeta_E + \sin\zeta_E)\}, \tag{4.14}$$

$$W_0 = -\tfrac{1}{2}\epsilon\{1 - \exp(-\zeta_e)(\cos\zeta_e + \sin\zeta_e) - \exp\zeta_E\cos\zeta_E\}, \tag{4.15}$$

with errors in V_0 and W_0 of order $\sqrt{\epsilon}\exp(-1/\sqrt{\epsilon})$ and $\epsilon\exp(-1/\sqrt{\epsilon})$, respectively.

It is interesting to note that, since $W_0 = O(\epsilon)$, equation (2.12) shows that the constant c_1 is of order $\sqrt{\nu}$ and hence by (2.9) the reduced pressure p is determined to a first approximation by the hydrostatic equation alone. Also, by the first term of (4.14) and by (3.21), if the centrifugal forces are small we can imagine the slope of the true free surface to be a generator of a dimensionless zonal velocity of $\frac{1}{2}$ which is cancelled out with depth by the thermal wind.

We now see that, since W_0 is of order ϵ, it is exactly the correct order of magnitude to ensure that F_1 and H_1 are of order unity in the core of the fluid. The solutions to (4.7) and (4.8) with the appropriate boundary conditions are

$$F_1 = \tfrac{1}{2}\sigma\{\zeta - 1 + \tfrac{1}{2}\sqrt{\epsilon} + \sqrt{\epsilon}\exp(-\zeta_e)\cos\zeta_e - \tfrac{1}{2}\sqrt{\epsilon}\exp\zeta_E(\cos\zeta_E + \sin\zeta_E)\},$$
$$H_1 = \sigma[\tfrac{1}{2}(\zeta^2 - 1) - \tfrac{1}{4}\sqrt{\epsilon}(\zeta^2 - 1) - \epsilon(\zeta - 1) + \tfrac{3}{4}\epsilon^{\frac{3}{2}} + \tfrac{1}{2}\epsilon^{\frac{3}{2}}\exp(-\zeta_e)\{\zeta_e(\sin\zeta_e - \cos\zeta_e)$$
$$+ 3\sin\zeta_e\} - \tfrac{1}{2}\epsilon\exp\zeta_E\sin\zeta_E - \tfrac{1}{2}\epsilon^{\frac{3}{2}}\exp\zeta_E\{\zeta_E\sin\zeta_E - \tfrac{3}{2}(\sin\zeta_E - \cos\zeta_E)\}],$$

with errors proportional to $\sqrt{\epsilon}\exp(-1/\sqrt{\epsilon})$ and $\epsilon\exp(-1/\sqrt{\epsilon})$ in F_1 and H_1 respectively. Again notice the difference in the orders of magnitude of the two components in the Ekman layers.

In discussing the first-order velocities we divide the flow field in the manner of §4.1. Then, in the core of the fluid, the largest terms of equation (4.9) are of order unity and are

$$2V_1' + \tfrac{1}{2}\zeta = \tfrac{1}{2}\sigma(\zeta - 1).$$

Define the solution to this equation to be V_1^T where

$$V_1^T = \tfrac{1}{8}\zeta^2(\sigma - 1) - \tfrac{1}{4}\sigma\zeta.$$

This zonal velocity, in contrast to that of §4.1, is thus driven by the non-linear terms which involve only the zero-order thermal wind $V_0^T = \frac{1}{2}\zeta$, and by the first-order temperature field. The surface stress condition (3.20) cannot be satisfied, however, and consequently we expect there to be a viscous component V_1^* which will be of order $\sqrt{\epsilon}$ and which will satisfy the boundary conditions that,

$$\text{at} \quad \zeta = 0, \quad V_1^* = 0 \quad \text{and, at} \quad \zeta = 1, \quad V_1^{*\prime} = \tfrac{1}{4}.$$

Associated with this zonal velocity, Ekman layer suction will produce a vertical velocity W_1, of order ϵ.

Thus write

$$V_1 = V_1^T + V_1^*, \quad F_1 = F_1^T + F_1^*,$$

where $F_1^T = \frac{1}{2}\sigma(\zeta - 1)$, and the equations (4.11) and (4.12) become

$$2V_1^{*\prime} + \zeta V_0^{*\prime} + V_0^* + 2V_0^* V_0^{*\prime} + \tfrac{1}{2}W_0 W_0''' = F_1^* + \tfrac{1}{2}\epsilon W_1^{\text{iv}},$$
$$-W_1' - \tfrac{1}{2}W_0'(\tfrac{1}{2}\zeta + V_0^*) + W_0(\tfrac{1}{2} + V_0^{*\prime}) = \tfrac{1}{4}\epsilon(\sigma - 1) + \epsilon V_1^{*\prime\prime},$$

where V_0^* is the viscous component of the zero-order zonal velocity. In the core the highest-order terms are

$$2V_1^{*\prime} + \zeta V_0^{*\prime} + V_0^* = F_1^*,$$
$$-W_1' - \tfrac{1}{2}W_0'\zeta + \tfrac{1}{2}W_0 = \tfrac{1}{4}\epsilon(\sigma - 1),$$

and we see that the terms in V_0^* and W_0 on the left-hand side are due to advection by V_0^T. Some of these terms appear in the upper boundary layer, where the highest-order terms are

$$(2V_{1,E}^{*\prime} + V_{0,E}^{*\prime})\sqrt{\epsilon} = \tfrac{1}{2}W_{1,E}^{\text{iv}}, \quad -W_{1,E}' - \tfrac{1}{2}W_{0,E}' = \sqrt{\epsilon}V_{1,E}^{*\prime\prime}.$$

Primes denote derivatives with respect to ζ_E. But in the lower boundary layer V_0^T is small and thus

$$2V_{1,e}'\sqrt{\epsilon} = \tfrac{1}{2}W_{1,e}^{\text{iv}}, \quad -W_{1,e}' = \sqrt{\epsilon}V_{1,e}'',$$

where primes denote derivatives with respect to ζ_e. The lower boundary layer to this order of magnitude is thus an Ekman layer.

The matching solutions to the equations are easily found

$$\left.\begin{aligned} V_{1,E}^* &= \tfrac{1}{2}\sqrt{\epsilon}\{1 + \tfrac{3}{4}\sigma + \tfrac{1}{4}\exp\zeta_E(\tfrac{1}{2}\cos\zeta_E + \tfrac{1}{2}\sin\zeta_E - \zeta_E\cos\zeta_E)\}, \\ V_1^* &= \tfrac{1}{4}\sqrt{\epsilon}\{(1 + \tfrac{1}{2}\sigma)\zeta + (1 + \sigma)\}, \\ V_{1,e}^* &= \tfrac{1}{4}\sqrt{\epsilon}(1 + \sigma)(1 - \exp(-\zeta_e)\cos\zeta_e); \end{aligned}\right\}$$

$$\left.\begin{aligned} W_{1,E} &= \tfrac{1}{4}\epsilon\{1 - \exp\zeta_E(\cos\zeta_E - \tfrac{1}{2}\zeta_E\cos\zeta_E + \tfrac{1}{2}\zeta_E\sin\zeta_E)\}, \\ W_1 &= \tfrac{1}{4}\epsilon\sigma(1 - \zeta) + \tfrac{1}{4}\epsilon, \\ W_{1,e} &= \tfrac{1}{4}\epsilon(1 + \sigma)\{1 - \exp(-\zeta_e)(\cos\zeta_e + \sin\zeta_e)\}. \end{aligned}\right\}$$

This process can be repeated for higher-order terms in ϵ until the errors are proportional to $\exp(-1/\sqrt{\epsilon})$.

It is clear from these expressions that the condition for the validity of the expansion procedure is $\sigma\beta \ll 1$ and thus conduction processes always dominate for a flow with low Rossby number.

We note in passing that, with both types of top boundary conditions on the velocity, the choice of a temperature maximum would simply reverse the linear zero-order flow. Also note that an alternative method of solution is to expand the variables simultaneously in powers of β and $\sigma\beta$.

5. Solutions for convection domination with a temperature minimum

5.1. *The singular regions*

We come now to the situation discussed in §3.3 when we expect convection processes to dominate in inviscid regions. Solutions of the equations (3.2)–(3.5) are posulated by the technique of expansion matching which has been described by Lagerstrom & Cole (1955) and Proudman & Pearson (1957). The details of the method and the manner in which it can be applied will be discussed in the course of the section.

Since the condition for convection domination involves both β and ϵ, it is more convenient to choose a new parameter λ which is defined by

$$\lambda = \epsilon\beta^{-2}. \tag{5.1}$$

Then clearly the thickness of the thermal layer is of order $\sqrt{\lambda}$. To simplify matters still further let us postulate a relationship between β and λ

$$\lambda = \beta^{\tau}, \tag{5.2}$$

where τ is some positive constant, the value of which will affect the solutions in some way.

From the discussion of § 3.3 it seems that the Ekman and thermal layers are singular regions of the equations of motion in that, in these layers, viscous and conduction terms respectively become significant and balance the Coriolis and convection terms. Since the structure of singular layers is determined solely by the equations of motion, it is possible to establish that these are the only types of singular layer by introducing a new co-ordinate

$$\xi = \zeta/\beta^{n} \quad (n \geqslant 0) \tag{5.3}$$

and by setting

$$\bar{V} = V, \quad \bar{W} = \sqrt{\epsilon}\, W = \beta^{1+\frac{1}{2}\tau} W, \quad \bar{F} = F, \quad \bar{H} = H, \tag{5.4}$$

where all the new variables are assumed to be of order unity. Then substitute (5.3) and (5.4) in the equations (3.2)–(3.5) and let β tend to zero for fixed ξ to obtain various limits of the equations for different values of n. Another reason for this procedure will be evident shortly. Thus we have

$$\left.\begin{aligned} &\text{limit } 1 \equiv n = 0, \\ &\qquad 2V' = F, \qquad W' = 0, \\ &-W'F + WF' = 0, \qquad WH' = 0, \end{aligned}\right\} \tag{5.5}$$

$$\left.\begin{aligned} &\text{limit } 2 \equiv 0 < n < \tfrac{1}{2}\tau, \\ &\qquad 2V' = 0, \qquad W' = 0, \\ &-W'F + WF' = 0, \qquad WH' = 0, \end{aligned}\right\} \tag{5.6}$$

$$\left.\begin{aligned} &\text{limit } 3 \equiv n = \tfrac{1}{2}\tau, \\ &\qquad 2V' = 0, \qquad W' = 0, \\ &\sigma(-W'F + WF') = F'', \qquad \sigma WH' = H'', \end{aligned}\right\} \tag{5.7}$$

$$\left.\begin{aligned} &\text{limit } 4 \equiv \tfrac{1}{2}\tau < n < 1 + \tfrac{1}{2}\tau, \\ &2V' = 0, \qquad W' = 0, \\ &0 = F'', \qquad 0 = H'', \end{aligned}\right\} \tag{5.8}$$

$$\left.\begin{aligned} &\text{limit } 5 \equiv n = 1 + \tfrac{1}{2}\tau, \\ &2V' = \tfrac{1}{2}W^{\text{iv}}, \qquad -W' = V'', \\ &0 = F'', \qquad 0 = H'', \end{aligned}\right\} \tag{5.9}$$

$$\left.\begin{aligned} &\text{limit } 6 \equiv n > 1 + \tfrac{1}{2}\tau, \\ &0 = W^{\text{iv}}, \qquad 0 = V'', \\ &0 = F'', \qquad 0 = H''. \end{aligned}\right\} \tag{5.10}$$

Primes denote derivatives with respect to ξ.

Now the relationship (5.3) with fixed ξ ensures that we are considering the flow in the neighbourhood of the boundary $\zeta = 0$. The different limits thus indicate, to a first approximation, the nature of the flow when $\zeta = O(\beta^{n})$. Clearly then, the equations (5.5) represent geostrophic flow with convection domination far from the boundaries. Conduction balances convection in (5.7) where

$$\zeta = O(\beta^{\frac{1}{2}\tau}) = O(\beta^{-1}\epsilon^{\frac{1}{2}}).$$

This is the thermal layer which we postulated in §3.3 and we confirm that in this layer the buoyancy forces no longer drive the zonal velocity so that V, as well as W, is constant. Finally, the equations (5.9) are those for an Ekman layer with heat transfer by conduction. The fact that W is constant everywhere outside this layer means that our assumption in (5.4) that $\overline{W}$ is of order $\epsilon^{\frac{1}{2}}$ is justified.

To apply the expansion matching method, the Ekman layer at the lower disk, say, is described by rewriting the full equations of motion in terms of the unit-order variables of (5.4) and the scaled co-ordinate ξ of (5.3) with $n = 1+\frac{1}{2}\tau$. It is then proposed that the exact solutions of the original equations of motion (3.2)–(3.5) can be expressed in that Ekman layer as asymptotic series in the small parameter β and the co-ordinate $\xi = \zeta/\beta^{1+\frac{1}{2}\tau}$. Moreover, it is postulated that these expansions are, in fact, the asymptotic solutions for the new, scaled, form of the equations of motion. Thus the expansions are valid in the domain of validity of the scaled equations. An exactly similar process is used for the thermal layer at the lower disk. If now the domains of validity of the equations which represent these thermal and Ekman layers should overlap, then in that region of overlap the corresponding expansions are both asymptotic representations of the exact solutions, and hence must be identical in some asymptotic sense which will be described in §5.2. This then provides the link between the expansions in the two regions and is the basis for the matching process. A similar procedure is carried out for the core and the upper thermal and Ekman layers.

Clearly the matching of the expansions depends on a knowledge of the domains of validity of the equations which describe the flow in each region, and it is for this reason that the intermediate limits (5.6), (5.8), and (5.10) have been included. It is a straightforward matter to show that limit 2 of the terms of equations (5.5) and (5.7), written in the original form of (3.2)–(3.5), gives equations (5.6). Thus, if $\zeta = O(\beta^n)$, $0 < n < \frac{1}{2}\tau$, equations (5.5) and (5.7) are equally valid for representing the flow. A similar analysis shows that equations (5.7) and (5.9) are equally valid in the range $\frac{1}{2}\tau < n < 1+\frac{1}{2}\tau$, and that equations (5.9) alone are valid when $n > 1+\frac{1}{2}\tau$. As we have seen, this overlap of the domains of validity of the various regions is absolutely essential to the matching process. The domains of validity are shown schematically in figure 4. From the diagram it is clear that the asymptotic series in the thermal layer provide the links between the series in the Ekman layer and the inviscid core.

Our model, however, is essentially one which depends on two parameters λ and β, and we must therefore examine the domains of validity in terms of these numbers so that the significance of the constant τ can be properly understood. Let the powers of β and λ in (5.5)–(5.10) be M and N, respectively. Then the critical values of n represented by limits 1, 3, and 5 are the critical values of $M+\tau N$. If, for example, $\tau = 2$, M and N satisfy the relationships for these limits

$$M+2N = 0, \quad M+2N = 1, \quad M+2N = 2,$$

respectively, and the domains of validity are determined by these lines in the (M, N) plane. These domains are shown schematically in figure 5, in which the origin represents the region far from the boundary. The significance of the constant τ is immediately apparent, for all the lines which bound the domains of

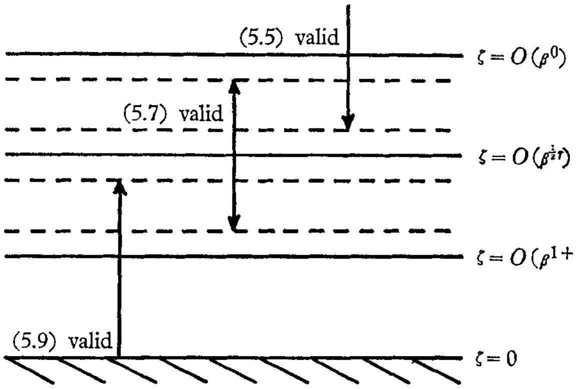

FIGURE 4. Domains of validity for a temperature minimum.

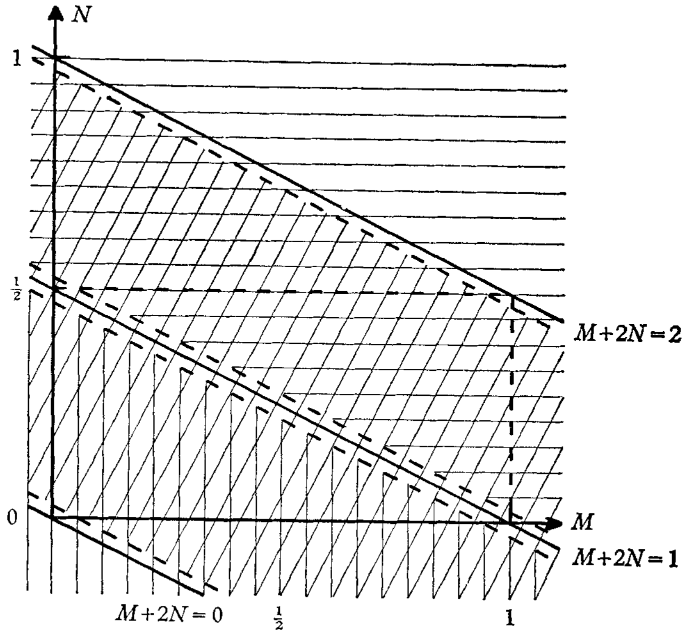

FIGURE 5. Domains of validity for a temperature minimum. The domains are represented by the type of shading: vertical, (5.5) valid; sloping, (5.7) valid; horizontal, (5.9) valid.

validity have slope $-1/\tau$. Thus, if the parameters λ and β are varied, the slope of each critical line is changed by the same amount, and the positions of the singular regions are not altered in relation to each other.

As τ is varied, we can see from figure 5 that the three critical lines always pass through one of the points $(0,0)$, $(0,\frac{1}{2})$, and $(1,\frac{1}{2})$. The structure of the singular regions, therefore, breaks down when τ tends to infinity or zero, for, in either of these limits, the middle critical line $M+\tau N = \frac{1}{2}\tau$ tends to coincide with one of the other two. It would appear then that there could be no overlap of the resulting two regions, and consequently no matching could take place. The explanation is that these two limiting values of τ imply that either β or λ is of order unity. If β is of order unity, the inertia terms of (3.2) and (3.3) are always of the same order of magnitude as the Coriolis terms, and the Ekman and thermal layers have the same thickness. On the other hand, if λ is of order unity, then by (5.1) the thermal layer covers the entire region between the disks and conduction can never be dominated by convection. Clearly, therefore, these two limiting values do not belong to our discussion.

The full equations in each region of the flow are a simple consequence of the above. In the core, the co-ordinate is ζ and the variables are $\bar{V} = V$, $\bar{W} = \beta^{1+\frac{1}{2}\tau}W$, $\bar{F} = F$, and $\bar{H} = H$. Thus

$$\left.\begin{aligned} \beta(2VV' + \tfrac{1}{2}\beta^{2+\tau}WW''') + 2V' &= F + \tfrac{1}{2}\beta^{3+\frac{3}{2}\tau}W^{\text{iv}}, \\ \beta(-W'V + WV') - W' &= \beta^{1+\frac{1}{2}\tau}V'', \\ \sigma(-W'F + WF') &= \beta^{\frac{1}{2}\tau}F'', \\ \sigma WH' &= \beta^{\frac{1}{2}\tau}(2F + H''). \end{aligned}\right\} \tag{5.11}$$

The equations for the thermal boundary layers are those obtained by limit 3 with $\zeta_t = \zeta/\beta^{\frac{1}{2}\tau}$ near the lower boundary and $\zeta_T = (\zeta - 1)/\beta^{\frac{1}{2}\tau}$ near the upper. Here the variables are

$$\bar{V} = V_t, \quad \bar{W} = \beta^{1+\frac{1}{2}\tau}W_t, \quad \bar{F} = F_t, \quad \bar{H} = H_t,$$

in the lower thermal layer, and the lower case subscripts are replaced by upper case in the corresponding upper layer:

$$\left.\begin{aligned} \beta(2V_tV_t' + \tfrac{1}{2}\beta^2W_tW_t''') + 2V_t' &= \beta^{\frac{1}{2}\tau}F_t + \tfrac{1}{2}\beta^3W_t^{\text{iv}}, \\ \beta(-W_t'V_t + W_tV_t') - W_t' &= \beta V_t'', \\ \sigma(-W_t'F_t + W_tF_t') &= F_t'', \\ \sigma W_tH_t' &= 2\beta^\tau F_t + H_t''. \end{aligned}\right\} \tag{5.12}$$

Lastly the Ekman layer equations correspond to limit 5 with co-ordinates $\zeta_e = \zeta/\beta^{1+\frac{1}{2}\tau} = \zeta_t/\beta$ near $\zeta = 0$, and $\zeta_E = (\zeta - 1)/\beta^{1+\frac{1}{2}\tau} = \zeta_T/\beta$ near $\zeta = 1$. Variables are denoted by lower and upper case letters respectively in the lower and upper layers:

$$\left.\begin{aligned} \beta(2V_eV_e' + \tfrac{1}{2}W_eW_e''') + 2V_e' &= \beta^{1+\frac{1}{2}\tau}F_e + \tfrac{1}{2}W_e^{\text{iv}}, \\ \beta(-W_e'V_e + W_eV_e') - W_e' &= V_e'', \\ \sigma\beta(-W_e'F_e + W_eF_e') &= F_e'', \\ \sigma\beta W_eH_e' &= 2\beta^{2+\tau}F_e + H_e''. \end{aligned}\right\} \tag{5.13}$$

5.2. *The matching process*

As we mentioned in §5.1, the solutions are obtained by expressing the variables in each region as asymptotic series:

$$\left.\begin{aligned} V(\zeta) &= \sum_0^\infty l_i(\beta)V_i(\zeta), \quad W(\zeta) = \sum_0^\infty m_i(\beta)W_i(\zeta), \\ F(\zeta) &= \sum_0^\infty n_i(\beta)F_i(\zeta), \quad H(\zeta) = \sum_0^\infty p_i(\beta)H_i(\zeta), \end{aligned}\right\} \tag{5.14}$$

where, for example,
$$\lim_{\beta\to 0}\frac{l_{i+1}(\beta)}{l_i(\beta)} = 0, \quad \text{and} \quad l_0(\beta) = 1.$$
Lower and upper case subscripts are added for the appropriate thermal and Ekman layers.

When the expansions (5.14) are inserted in their appropriate equations, it is clear from (5.5) to (5.10) that the value of τ does not affect the zero-order solutions. But, by (5.11)–(5.13), the value of τ determines the $l_i(\beta)$, etc., and we shall see that as τ varies there is, in fact, an infinite number of solutions.

Only the expansions in the two Ekman layers can satisfy any boundary conditions, these being either (3.6) or (3.7). All other expansions are determined by a matching process which, as we noted in §5.1, depends on the fact that each series is a locally valid expansion of the exact solutions of the equations of motion. Then, since the domains of validity of two neighbouring expansions overlap, their intermediate limits must be identical. For example
$$\text{limit 2 of } V - V_t \equiv 0 \tag{5.15}$$
equates the zero-order terms of the core and thermal layer expansions of the zonal velocity. Matching of higher-order terms is achieved by subtracting the appropriate partial sum from each expansion, before the limit is taken. Without the region of overlap, therefore, matching would be impossible.

Since we know that matching is possible, the procedure of (5.15), etc., can be simplified by stating, for example, that the asymptotic expansion of $\Sigma l_{t,i}(\beta)V_{t,i}(\zeta_t)$ for large ζ_t, expressed in terms of $\zeta = \beta^{\frac{1}{2}\tau}\zeta_t$, must be identical with the asymptotic expansion of $\Sigma l_i(\beta)V_i(\zeta)$ for small ζ. Thus terms of the form $\exp(-\zeta/\beta^{\frac{1}{2}\tau})$ in the lower thermal layer will not have a counterpart in the core. This procedure is legitimate although it implies that the expansion for V_t is valid when ζ is of order β^0. Alternatively, matching can be achieved by equating the asymptotic expansion of $\Sigma l_i(\beta)V_i(\zeta)$ for small ζ, expressed in terms of ζ_t, to the asymptotic expansion of $\Sigma l_{t,i}(\beta)V_{t,i}(\zeta_t)$ for large ζ_t.

In practice of course it is necessary to solve for one term at a time. The equations for the first few terms in each series are very simple, and our procedure will be to solve all the zero-order terms at once, then the first-order and so on.

5.3. *The zero-order terms*

The zero-order terms, in order from the top Ekman layer downwards, are
$$\left.\begin{aligned} 2V'_{E,0} &= \tfrac{1}{2}W^{\text{iv}}_{E,0} & -W'_{E,0} &= V''_{E,0}, \\ 0 &= F''_{E,0}, & 0 &= H''_{E,0}, \end{aligned}\right\} \tag{5.16}$$
$$\left.\begin{aligned} 2V'_{T,0} &= 0, & W'_{T,0} &= 0, \\ \sigma W_{T,0}F'_{T,0} &= F''_{T,0}, & \sigma W_{T,0}H'_{T,0} &= H''_{T,0}, \end{aligned}\right\} \tag{5.17}$$
$$\left.\begin{aligned} 2V'_0 &= F_0, & W'_0 &= 0, \\ W_0F'_0 &= 0, & W_0H'_0 &= 0, \end{aligned}\right\} \tag{5.18}$$
$$\left.\begin{aligned} 2V'_{t,0} &= 0, & W'_{t,0} &= 0, \\ \sigma W_{t,0}F'_{t,0} &= F''_{t,0}, & \sigma W_{t,0}H'_{t,0} &= H''_{t,0}, \end{aligned}\right\} \tag{5.19}$$

$$\left.\begin{aligned} 2V'_{e,0} &= \tfrac{1}{2}W^{\text{iv}}_{e}, & -W'_{e,0} &= V''_{e,0}, \\ 0 &= F''_{e,0}, & 0 &= H''_{e,0}, \end{aligned}\right\} \tag{5.20}$$

with boundary conditions by (3.6) and (3.7) that,

$$\text{at} \quad \zeta_E = 0, \quad V_{E,0} = W_{E,0} = W'_{E,0} = H_{E,0} = 0, \quad F_{E,0} = 1, \tag{5.21}$$

$$\text{at} \quad \zeta_e = 0, \quad V_{e,0} = W_{e,0} = W'_{e,0} = F'_{e,0} = H'_{e,0} = 0. \tag{5.22}$$

As we deduced in §3.3, since the zero-order vertical velocity is constant outside the Ekman layers, the solutions for the temperature components in the thermal layers are exponentials. But only the layer into which the fluid flows from the core can admit a bounded solution, and consequently only one thermal layer can exist to this order for any given boundary conditions. An examination of equations (5.12) shows that, in the region where there is no thermal layer in the zero-order terms, the zero-order vertical velocity determines the exponential behaviour of the first-order temperature field, but the sign of this velocity is such that the exponential is unbounded. Hence only one thermal layer can exist to all orders.

To solve equations (5.16)–(5.19) we assume that there is a downward flux of fluid outside the Ekman layers. Matching solutions are easily obtained:

$$\left.\begin{aligned} &V_{E,0} = \tfrac{1}{4}(1-\exp\zeta_E\cos\zeta_E), \\ &V_{T,0} = \tfrac{1}{4}, \quad V_0 = \tfrac{1}{4}(2\zeta-1), \quad V_{t,0} = -\tfrac{1}{4}, \\ &V_{e,0} = -\tfrac{1}{4}(1-\exp(-\zeta_e)\cos\zeta_e); \end{aligned}\right\} \tag{5.23}$$

$$\left.\begin{aligned} &W_{E,0} = -\tfrac{1}{4}\{1-\exp\zeta_E(\cos\zeta_E-\sin\zeta_E)\}, \\ &W_{T,0} = W_0 = W_{t,0} = -\tfrac{1}{4}, \\ &W_{e,0} = -\tfrac{1}{4}\{1-\exp(-\zeta_e)(\cos\zeta_e+\sin\zeta_e)\}; \end{aligned}\right\} \tag{5.24}$$

$$\left.\begin{aligned} &F_{E,0} = F_{T,0} = F_0 = 1, \\ &F_{t,0} = a_1\exp(-\tfrac{1}{4}\sigma\zeta_t)+1, \\ &F_{e,0} = 1+a_1; \end{aligned}\right\} \tag{5.25}$$

$$\left.\begin{aligned} &H_{E,0} = H_{T,0} = H_0 = 0, \\ &H_{t,0} = a_2\exp(-\tfrac{1}{4}\sigma\zeta_t), \quad H_{e,0} = a_2; \end{aligned}\right\} \tag{5.26}$$

where a_1 and a_2 are constants. The streamlines of the secondary flow are thus very similar to those of figure 2.

The two constants must be found by matching the coefficients of ζ_t in (5.25) on to terms of the form $\beta\zeta_e$ in F_e and H_e. Since the value of τ is unknown, the coefficient of β in F_e and H_e may not necessarily be the next term in the expansion. Let it correspond to the jth term so that $n_{e,j} = p_{e,j} = \beta$. The equations for $F_{e,j}$ and $H_{e,j}$ are then by (5.13)

$$\sigma(-W'_{e,0}F_{e,0}+W_{e,0}F'_{e,0}) = F''_{e,j}, \quad \sigma W_{e,0}H'_{e,0} = H''_{e,j},$$

with boundary conditions by (3.6) that,

$$\text{at} \quad \zeta_e = 0, \quad F'_{e,j} = H'_{e,j} = 0.$$

Thc solutions are

$$F_{e,j} = \tfrac{1}{4}\sigma(1+a_1)(\zeta_e+\exp(-\zeta_e)\cos\zeta_e)+b_1, \quad H_{e,j} = b_2, \tag{5.27}$$

where b_1 and b_2 are constants. Here again, as in §4.1, we see that F and H behave in a different manner in the Ekman layers because the former is convected radially and vertically whereas the latter is convected only vertically. Matching the solutions in the thermal and Ekman layers we have

$$a_1 = -\tfrac{1}{2}, \quad a_2 = 0.$$

Thus (5.25) and (5.26) are replaced by

$$\begin{aligned} F_{E,0} &= F_{T,0} = F_0 = 1, \\ F_{t,0} &= 1-\tfrac{1}{2}\exp(-\tfrac{1}{4}\sigma\zeta_t), \quad F_{e,0} = \tfrac{1}{2}, \\ H_{E,0} &= H_{T,0} = H_0 = H_{t,0} = H_{e,0} = 0, \end{aligned}$$

which confirms the deduction (3.14).

The calculation of $F_{e,j}$ also confirms the assumption of a negative vertical velocity. For, if the vertical velocity is positive, there can be no lower thermal layer and thus a_1 vanishes and $F_{e,0}$ and $F_{t,0}$ are constants. The relation (5.27) is then

$$F_{e,j} = \tfrac{1}{4}\sigma F_{e,0}(\zeta_e+\exp(-\zeta_e)\cos\zeta_e)+b_1,$$

the linear term of which clearly cannot match any term in the thermal layer. Thus $F_{e,0} = 0$, but this in turn implies a paradox in the matching of the zonal velocities. The vertical velocity therefore must be negative. Moreover, since the vertical velocity can never be positive we have confirmed our deductions of §3.4 that the problem is not properly posed if we expect an up-draught from an insulated lower disk to take place with convection domination.

5.4. *The first-order terms*

As we noted in §5.2 the precise value of τ now begins to affect the expansions by determining the order of the l_i, etc., in the asymptotic series. An examination of the equations (5.11)–(5.13) shows that the value $\tau = 2$, i.e. $\delta_\tau = \beta$, separates two types of solutions in the first-order terms.

If $\tau > 2$, the first-order terms are generated by the non-linear inertia and convection terms throughout the field of flow. It seems reasonable, then, to assume that all the l_1, m_1, etc, are simply β. Thus the first-order equations with the simplest zero-order solutions substituted are

$$\left.\begin{aligned} 2V_{E,0}V'_{E,0}+\tfrac{1}{2}W_{E,0}W'''_{E,0}+2V'_{E,1} &= \tfrac{1}{2}W^{\text{iv}}_{E,1}, \\ -W'_{E,0}V_{E,0}+W_{E,0}V'_{E,0}-W'_{E,1} &= V''_{E,1}, \\ -\sigma W'_{E,0} = F''_{E,1}, \quad 0 &= H''_{E,1}, \end{aligned}\right\} \tag{5.28}$$

$$\left.\begin{aligned} 2V'_{T,1} = 0 \qquad & -W'_{T,1} = 0, \\ -\tfrac{1}{4}\sigma F'_{T,1} = F''_{T,1}, \qquad & -\tfrac{1}{4}\sigma H'_{T,1} = H''_{T,1}, \end{aligned}\right\} \tag{5.29}$$

$$\left.\begin{aligned} \tfrac{1}{4}(2\zeta-1)+2V'_1 = F_1, \qquad & -\tfrac{1}{8}-W'_1 = 0, \\ -W'_1-\tfrac{1}{4}F'_1 = 0, \qquad & -\tfrac{1}{4}H'_1 = 0, \end{aligned}\right\} \tag{5.30}$$

$$\left.\begin{aligned} 2V'_{t,1} &= 0, \qquad W'_{t,1} = 0, \\ \sigma(-\tfrac{1}{4}F'_{t,1}+\tfrac{1}{8}W_{t,1}\sigma\exp(-\tfrac{1}{4}\sigma\zeta_t)) &= F''_{t,1}, \quad -\tfrac{1}{4}\sigma H'_{t,1} = H''_{t,1}, \end{aligned}\right\} \tag{5.31}$$

$$\left.\begin{aligned} 2V'_{e,0}V_{e,0}+\tfrac{1}{2}W_{e,0}W'''_{e,0}+2V'_{e,1} &= \tfrac{1}{2}W^{\text{iv}}_{e,1}, \\ -W'_{e,0}V_{e,0}+W_{e,0}V'_{e,0}-W'_{e,1} &= V''_{e,1}, \\ -\tfrac{1}{2}\sigma W'_{e,0} = F''_{e,1}, \quad 0 &= H''_{e,1}. \end{aligned}\right\} \tag{5.32}$$

The boundary conditions are similar to (5.21) and (5.22) except that

$$F_{e,1} = 0. \tag{5.33}$$

The last two equations of (5.32) have, of course, been solved in §5.3. Once again two arbitrary constants appear in the temperature solutions in the lower thermal and Ekman layers, and they are determined in a like manner to a_1 and a_2 by resort to the coefficients of β^2 in F_e and H_e. The matching solutions are obtained in a straightforward manner:

$$V_{E,1} = \tfrac{1}{80}+\tfrac{1}{16}\sigma(1-\exp\zeta_E\cos\zeta_E)+\tfrac{1}{160}\exp\zeta_E(5\zeta_E\sin\zeta_E-2\exp\zeta_E-\sin\zeta_E),$$

$$V_{T,1} = \tfrac{1}{16}\sigma+\tfrac{1}{80}, \quad V_1 = \tfrac{1}{8}\zeta(\sigma-1)+\tfrac{1}{16}(\tfrac{11}{5}-\sigma), \quad V_{t,1} = \tfrac{1}{16}(\tfrac{11}{5}-\sigma),$$

$$V_{e,1} = \tfrac{11}{80}-\tfrac{1}{16}\sigma(1-\exp(-\zeta_e)\cos\zeta_e)$$
$$+\tfrac{1}{160}\exp(-\zeta_e)(5\zeta_e\sin\zeta_e-2\exp(-\zeta_e)-20\cos\zeta_e+\sin\zeta_e),$$

$$W_{E,1} = -\tfrac{1}{16}\sigma\{1-\exp\zeta_E(\cos\zeta_E-\sin\zeta_E)\}$$
$$+\tfrac{1}{160}\exp\zeta_E\{\cos\zeta_E+6\sin\zeta_E-\exp\zeta_E-5\zeta_E(\cos\zeta_E+\sin\zeta_E)\},$$

$$W_{T,1} = -\tfrac{1}{16}\sigma, \quad W_1 = \tfrac{1}{8}(1-\tfrac{1}{2}\sigma-\zeta), \quad W_{t,1} = \tfrac{1}{16}(2-\sigma),$$

$$W_{e,1} = -\tfrac{1}{16}\sigma\{1-\exp(-\zeta_e)(\cos\zeta_e+\sin\zeta_e)\}+\tfrac{1}{8}$$
$$+\tfrac{1}{160}\exp(-\zeta_e)\{\exp(-\zeta_e)-21\cos\zeta_e-14\sin\zeta_e-5\zeta_e(\cos\zeta_e-\sin\zeta_e)\},$$

$$F_{E,1} = \tfrac{1}{4}\sigma(1-\exp\zeta_E\cos\zeta_E), \quad F_{T,1} = \tfrac{1}{4}\sigma, \quad F_1 = \tfrac{1}{2}\zeta+\tfrac{1}{4}\sigma-\tfrac{1}{2},$$

$$F_{t,1} = \tfrac{1}{4}\sigma-\tfrac{1}{2}+\exp(-\tfrac{1}{4}\sigma\zeta_t)\{\tfrac{1}{4}-\tfrac{9}{32}\sigma-\tfrac{1}{32}\sigma\zeta_t(2-\sigma)\},$$

$$F_{e,1} = \tfrac{1}{8}\sigma(\zeta_e+\exp(-\zeta_e)\cos\zeta_e)-\tfrac{1}{4}(1+\tfrac{1}{8}\sigma),$$

$$H_{E,1} = H_{T,1} = H_1 = H_{t,1} = H_{e,1} = 0.$$

If $\tau < 2$, only the convection terms outside the Ekman layers, and the geostrophic balance in the thermal layers, can generate first-order terms. All the l_1, etc., must be equal to $\beta^{\frac{1}{2}\tau}$ and the first-order equations are

$$\begin{aligned} 2V'_{E,1} &= \tfrac{1}{2}W^{\text{iv}}_{E,1}, & -W'_{E,1} &= V''_{E,1}, \\ 0 &= F''_{E,1}, & 0 &= H''_{E,1}, \\ 2V'_{T,1} &= 1, & W'_{T,1} &= 0, \\ -\tfrac{1}{4}\sigma F'_{T,1} &= F''_{T,1}, & -\tfrac{1}{4}\sigma H'_{T,1} &= H''_{T,1}, \\ 2V'_1 &= F_1, & W'_1 &= 0, \\ F'_1 &= 0, & -\tfrac{1}{4}\sigma H'_1 &= 2, \\ 2V'_{t,1} &= 1-\tfrac{1}{2}\exp(-\tfrac{1}{4}\sigma\zeta_t), & W'_{t,1} &= 0, \\ \sigma(-\tfrac{1}{4}F'_{t,1}+W_{t,1}F'_{t,0}) &= F''_{t,1}, & -\tfrac{1}{4}\sigma H'_{t,1} &= H''_{t,1}, \\ 2V'_{e,1} &= \tfrac{1}{2}W''_{e,1}, & -W'_{e,1} &= V''_{e,1}, \\ 0 &= F''_{e,1}, & 0 &= H''_{e,1}, \end{aligned}$$

with the boundary conditions as in (5.33). This is the first occasion on which H has a non-zero component, and, in fact, the coefficients of $\beta^{\frac{1}{2}\tau}$ are the first possible values of H, whatever the value of τ, because H must be generated by F. The arbitrary constants which appear in the temperature components are this time found by the coefficients of $\beta^{1+\frac{1}{2}\tau}$ in F_e and H_e.

The solutions are

$$
\begin{aligned}
V_{E,1} &= -\tfrac{1}{2}\sigma^{-1}(1-\exp\zeta_E\cos\zeta_E), \quad V_{T,1} = -\tfrac{1}{2}\sigma^{-1}+\tfrac{1}{2}\zeta_T, \quad V_1 = -\tfrac{1}{2}\sigma^{-1},\\
V_{t,1} &= -\tfrac{1}{2}\sigma^{-1}+\tfrac{1}{2}\zeta_t+\sigma^{-1}\exp(-\tfrac{1}{4}\sigma\zeta_t),\\
V_{e,1} &= -\tfrac{1}{2}\sigma^{-1}(1-\exp(-\zeta_e)\cos\zeta_e),\\
W_{E,1} &= \tfrac{1}{2}\sigma^{-1}\{1-\exp\zeta_E(\cos\zeta_E-\sin\zeta_E)\},\\
W_{T,1} &= W_1 = W_{t,1} = \tfrac{1}{2}\sigma^{-1},\\
W_{e,1} &= \tfrac{1}{2}\sigma^{-1}\{1-\exp(-\zeta_e)(\cos\zeta_e+\sin\zeta_e)\},\\
F_{E,1} &= F_{T,1} = F_1 = 0, \quad F_{t,1} = -\tfrac{1}{4}\zeta_t\exp(-\tfrac{1}{4}\sigma\zeta_t), \quad F_{e,1} = 0,\\
H_{E,1} &= H_{T,1} = 0, \quad H_1 = 8(1-\zeta)/\sigma, \quad H_{t,1} = H_{e,1} = 8/\sigma.
\end{aligned}
$$

Since the equations for the first-order variables are always linear, if $\tau = 2$, then the resulting solutions will simply be the sum of the two sets of solutions for $\tau > 2$ and $\tau < 2$, and will be the coefficients of β.

6. Convection domination with a temperature maximum

6.1. *The singular regions*

In this section the situation which was described in §3.4 is considered. Thus we redefine the temperature scale by (3.15) and use the boundary conditions (3.16) on $\zeta = 1$.

A similar analysis to that of §5.1 confirms the predictions of §3.4 and determines the relevant equations of motion in the different regions of flow. In terms of λ and β the thermal layer is of thickness $\delta'_T = \lambda^{\frac{1}{4}} = \beta^{\frac{1}{4}\tau}$, which is also the order of magnitude of the zonal velocity. The unit-order variables are defined by

$$\bar{V} = \beta^{\frac{1}{4}\tau}V, \quad \bar{W} = \beta^{1+\frac{3}{4}\tau}W, \quad \bar{F} = F, \quad \bar{H} = H,$$

and the core equations become

$$
\left.
\begin{aligned}
\beta^{1+\frac{1}{2}\tau}(2VV'+\tfrac{1}{2}\beta^{2+\tau}WW''')+2\beta^{\frac{1}{2}\tau}V' &= F+\tfrac{1}{2}\beta^{3+\frac{3}{2}\tau}W^{\text{iv}},\\
\beta^{1+\frac{1}{2}\tau}(-W'V+WV')-W' &= \beta^{1+\frac{1}{2}\tau}V'',\\
\sigma(-W'F+WF') &= \beta^{\frac{1}{2}\tau}F'',\\
\sigma WH' &= \beta^{\frac{1}{2}\tau}(2F+H'').
\end{aligned}
\right\}
\qquad (6.1)
$$

The thermal layer co-ordinates are now $\zeta_t = \zeta/\beta^{\frac{1}{4}\tau}$ and $\zeta_T = (\zeta-1)/\beta^{\frac{1}{4}\tau}$ to give the equations

$$
\left.
\begin{aligned}
\beta^{1+\frac{1}{4}\tau}(2V_tV'_t+\tfrac{1}{2}\beta^{2+\frac{1}{2}\tau}W_tW'''_t)+2V'_t &= F_t+\tfrac{1}{2}\beta^{3+\frac{3}{4}\tau}W^{\text{iv}}_t,\\
\beta^{1+\frac{1}{4}\tau}(-W'_tV_t+W_tV'_t)-W'_t &= \beta^{1+\frac{1}{4}\tau}V''_t,\\
\sigma(-W'_tF_t+W_tF'_t) &= F''_t,\\
\sigma W_tH'_t &= 2\beta^{\frac{1}{2}\tau}F_t+H''_t.
\end{aligned}
\right\}
\qquad (6.2)
$$

Finally, the Ekman layers have again the co-ordinates $\zeta_e = \zeta/\beta^{1+\frac{1}{2}\tau}$ and $\zeta_E = (\zeta-1)/\beta^{1+\frac{1}{2}\tau}$ with the equations

$$\left.\begin{aligned} \beta^{1+\frac{1}{2}\tau}(2V_e V_e' + \tfrac{1}{2}W_e W_e''') + 2V_e' &= \beta^{1+\frac{1}{2}\tau}F_e + \tfrac{1}{2}W_e^{\text{iv}}, \\ \beta^{1+\frac{1}{2}\tau}(-W_e' V_e + W_e V_e') - W_e' &= V_e'', \\ \sigma\beta^{1+\frac{1}{2}\tau}(-W_e' F_e + W_e F_e') &= F_e'', \\ \sigma\beta^{1+\frac{1}{2}\tau} W_e H_e' &= 2\beta^{2+\tau}F_e + H_e''. \end{aligned}\right\} \tag{6.3}$$

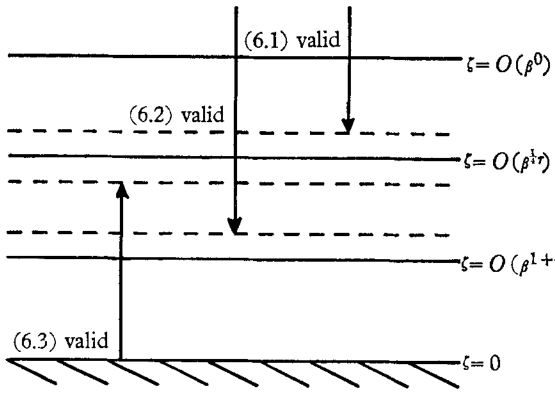

FIGURE 6. Domains of validity for a temperature maximum.

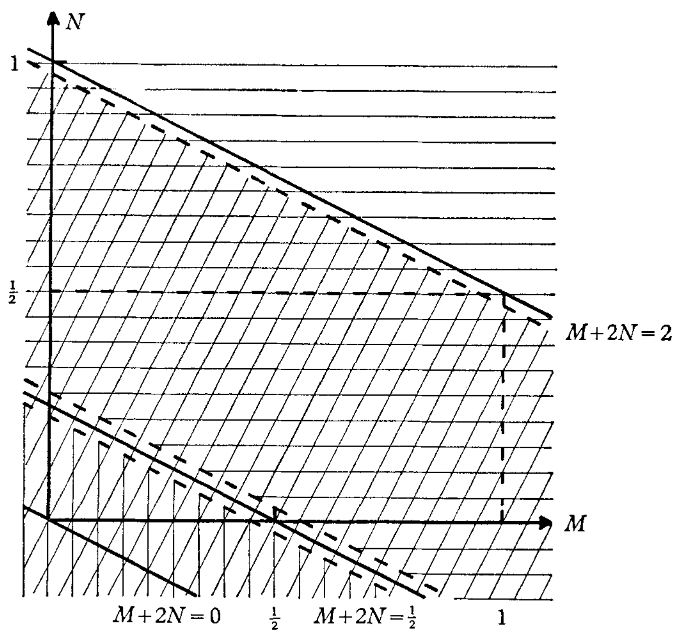

FIGURE 7. Domains of validity for a temperature maximum. The domains are represented by the type of shading: vertical, (6.1) valid; sloping, (6·2) valid; horizontal, (6.3) valid.

Figures 6 and 7 show the domains of validity of these three sets of equations. Since the equations for the thermal layers are valid in the core, only the equations (6.2) and (6.3) would be required in the situation of one disk in a semi-infinite fluid. The presence of the other disk, however, means that the core equations (6.1) are necessary in order to link the two strained co-ordinate systems near $\zeta = 0$ and $\zeta = 1$.

In figure 7 we can see that, as τ varies, the critical lines always pass through one of the points (0, 0), (0, $\frac{1}{4}$), and (1, $\frac{1}{2}$). Thus, in contrast to § 5.1, as τ tends to either zero or infinity the matching process is unaffected, for in either limit the domain of validity of (6.2) overlaps the domains of (6.1) and (6.3). The former limit implies that λ is of order unity, and thus the thermal layer covers the field of flow. But we observed in the last paragraph that the equations of the thermal layer are valid in the core, and hence this limit does not change the structure of the three regions. The other limit implies that β is of order unity. An examination of equations (6.1)–(6.3), however, shows that, since the velocity scales depend on λ, the thermal Rossby number β is not the true Rossby number of the motion; the ratio of inertia to Coriolis terms is everywhere of order $\beta^{1+\frac{1}{4}\tau} = \beta\lambda^{\frac{1}{4}}$. Thus the analysis holds if it is only the parameter ϵ which is small.

Let the solutions be expressed as the asymptotic series (5.14).

6.2. *The solutions*

The zero-order equations are

$$\left.\begin{aligned} 2V'_{E,0} &= \tfrac{1}{2}W^{\text{iv}}_{E,0}, & -W'_{E,0} &= V''_{E,0}, \\ 0 &= F''_{E,0}, & 0 &= H''_{E,0}, \end{aligned}\right\} \tag{6.4}$$

$$\left.\begin{aligned} 2V'_{T,0} &= F_{T,0}, & -W'_{T,0} &= 0, \\ \sigma W_{T,0}F'_{T,0} &= F''_{T,0}, & \sigma W_{T,0}H'_{T,0} &= H''_{T,0}, \end{aligned}\right\} \tag{6.5}$$

$$\left.\begin{aligned} 0 &= F_0, & W'_0 &= 0, \\ W_0F'_0 &= 0, & W_0H'_0 &= 0, \end{aligned}\right\} \tag{6.6}$$

$$\left.\begin{aligned} 2V'_{t,0} &= F_{t,0}, & -W'_{t,0} &= 0, \\ \sigma W_{t,0}F'_{t,0} &= F''_{t,0}, & \sigma W_{t,0}H'_{t,0} &= H''_{t,0}, \end{aligned}\right\} \tag{6.7}$$

$$\left.\begin{aligned} 2V'_{e,0} &= \tfrac{1}{2}W^{\text{iv}}_{e,0}, & -W'_{e,0} &= V''_{e,0}, \\ 0 &= F''_{e,0}, & 0 &= H''_{e,0}, \end{aligned}\right\} \tag{6.8}$$

with the boundary conditions from (3.6), (3.7), (3.16) and (3.17),

$$\text{at} \quad \zeta_E = 0, \quad W_{E,0} = W'_{E,0} = V_{E,0} = 0, \quad F_{E,0} = -1, \quad H_{E,0} = 0, \tag{6.9}$$

$$\text{at} \quad \zeta_e = 0, \quad W_{e,0} = W'_{e,0} = V_{e,0} = F_{e,0} = 0, \quad H_{e,0} = h_0. \tag{6.10}$$

An immediate deduction from these equations is that the vertical velocity must be positive for, as we have seen, a down-draught carries the non-uniform temperature on the upper disk into the core, where (6.6) shows that the fluid must be homogeneous.

As they stand, the equations do not have a solution, for the form of V_0 in the core is not specified. This, by (6.1), is determined by coefficients of $\beta^{\frac{1}{4}\tau}$, but it is a simple matter to show that below the upper thermal layer, to this order, the radial component of the temperature field vanishes. The zero-order velocity in the core is therefore constant as we had expected in §3.4 and the solutions are easily obtained:

$$
\begin{aligned}
V_{E,0} &= -\tfrac{1}{2}\sigma^{-\frac{1}{2}}(1-\exp\zeta_E\cos\zeta_E),\\
V_{T,0} &= \tfrac{1}{2}\sigma^{-\frac{1}{2}} - \sigma^{-\frac{1}{2}}\exp(\tfrac{1}{2}\sigma\zeta_T),\\
V_0 = V_{t,0} &= \tfrac{1}{2}\sigma^{-\frac{1}{2}},\\
V_{e,0} &= \tfrac{1}{2}\sigma^{-\frac{1}{2}}(1-\exp(-\zeta_e)\cos\zeta_e);
\end{aligned}
$$

$$
\begin{aligned}
W_{E,0} &= \tfrac{1}{2}\sigma^{-\frac{1}{2}}\{1-\exp\zeta_E(\cos\zeta_E-\sin\zeta_E)\},\\
W_{T,0} &= W_0 = W_{t,0} = \tfrac{1}{2}\sigma^{-\frac{1}{2}},\\
W_{e,0} &= \tfrac{1}{2}\sigma^{-\frac{1}{2}}\{1-\exp(-\zeta_e)(\cos\zeta_e+\sin\zeta_e)\};
\end{aligned}
$$

$$
\begin{aligned}
F_{E,0} &= -1, \quad F_{T,0} = -\exp(\tfrac{1}{2}\sqrt{\sigma}\zeta_T),\\
F_0 &= F_{t,0} = F_{e,0} = 0;
\end{aligned}
$$

$$
\begin{aligned}
H_{E,0} &= 0, \quad H_{T,0} = h_0(1-\exp(\tfrac{1}{2}\sqrt{\sigma}\,\zeta_T)),\\
H_0 &= H_{t,0} = H_{e,0} = h_0.
\end{aligned}
$$

Thus below the upper thermal layer the fluid is at a uniform temperature h_0 to a first approximation. An examination of the next and higher-order terms in the energy equations in those lower regions shows that this must be true to all orders. Thus we can safely omit all the temperature terms in the core and lower Ekman layers.

The first-order terms in V, W, and F are clearly generated by the non-linear inertia and convection terms of the Ekman and thermal layers and are coefficients of $\beta^{1+\frac{1}{4}\tau}$. But the equations (6.2) indicate that there is a coefficient of $\beta^{\frac{1}{2}\tau}$ in H, generated by F_0, in the thermal layer. Thus $\tau = 4$ separates two different types of first-order solution, and this value, as in §5.4, corresponds to a thermal layer of thickness $\delta_T' = \beta$.

If $\tau > 4$, all the l_1, etc., are equal to $\beta^{1+\frac{1}{4}\tau}$ to give the equations

$$
\left.
\begin{aligned}
2V_{E,0}V'_{E,0}+\tfrac{1}{2}W_{E,0}W'''_{E,0}+2V'_{E,1} &= F_{E,0}+\tfrac{1}{2}W^{\text{iv}}_{E,1},\\
-W'_{E,0}V_{E,0}+W_{E,0}V'_{E,0}-W'_{E,1} &= V''_{E,1},\\
\sigma W'_{E,0} = F''_{E,1}, \quad 0 &= H''_{E,1};
\end{aligned}
\right\} \tag{6.11}
$$

$$
\left.
\begin{aligned}
2V_{T,0}V'_{T,0}+2V'_{T,1} &= F_{T,1},\\
\tfrac{1}{2}\sigma^{-\frac{1}{2}}V'_{T,0}-W'_{T,1} &= V''_{T,0}\\
\sigma(-W'_{T,1}F_{T,0}+\tfrac{1}{2}\sigma^{-\frac{1}{2}}F'_{T,1}+W_{T,1}F'_{T,0}) &= F''_{T,1},\\
\sigma(W_{T,1}H'_{T,0}+\tfrac{1}{2}\sigma^{-\frac{1}{2}}H'_{T,1}) &= H''_{T,1};
\end{aligned}
\right\} \tag{6.12}
$$

$$V'_1 = W'_1 = 0; \tag{6.13}$$

$$V'_{t,1} = W'_{t,1} = 0; \tag{6.14}$$

$$
\left.
\begin{aligned}
2V'_{e,0}V_{e,0}+\tfrac{1}{2}W_{e,0}W'''_{e,0}+2V'_{e,1} &= \tfrac{1}{2}W^{\text{iv}}_{e,1},\\
-W'_{e,0}V_{e,0}+W_{e,0}V'_{e,0}-W'_{e,1} &= V''_{e,1}.
\end{aligned}
\right\} \tag{6.15}
$$

Actually, the equation for V_1, (6.13), has been obtained from coefficients of $\beta^{1+\frac{1}{2}\tau}$.

The boundary conditions are:

$$\text{at} \quad \zeta_E = 0, \quad W_{E,1} = W'_{E,1} = V_{E,1} = F_{E,1} = H_{E,1} = 0;$$
$$\text{at} \quad \zeta_e = 0, \quad W_{e,1} = W'_{e,1} = V_{e,1} = 0;$$

and in addition, as $\zeta_T \to -\infty$, $F_{T,1}, H_{T,1} \to 0$.

A routine calculation yields the solutions:

$$\begin{aligned}
V_{E,1} &= \tfrac{17}{40}\sigma^{-1} - \tfrac{1}{4} - \tfrac{1}{2}\zeta_E \\
&\quad + \exp\zeta_E\{\tfrac{1}{40}\sigma^{-1}(5\zeta_E - 1)\sin\zeta_E - \tfrac{1}{8}(3\sigma^{-1} - 2)\cos\zeta_E - \tfrac{1}{20}\sigma^{-1}\exp\zeta_E\}, \\
V_{T,1} &= \exp(\tfrac{1}{2}\sqrt{\sigma}\,\zeta_T)\{\tfrac{3}{4}\sigma^{-1} + (\zeta_T/\sqrt{\sigma})(\tfrac{1}{4}\sigma - \tfrac{1}{8})\} - \tfrac{1}{2}\sigma^{-1}\exp(\sqrt{\sigma}\,\zeta_T) + \tfrac{7}{40}\sigma^{-1} - \tfrac{1}{4}, \\
V_1 &= V_{t,1} = \tfrac{7}{40}\sigma^{-1} - \tfrac{1}{4}, \\
V_{e,1} &= \tfrac{7}{40}\sigma^{-1} - \tfrac{1}{4} + \exp(-\zeta_e)\{\tfrac{1}{40}\sigma^{-1}(5\zeta_e + 1)\sin\zeta_e - (\tfrac{1}{8}\sigma^{-1} - \tfrac{1}{4})\cos\zeta_e \\
&\quad - \tfrac{1}{20}\sigma^{-1}\exp(-\zeta_e)\};
\end{aligned}$$

$$\begin{aligned}
W_{E,1} &= -\tfrac{1}{8}(3/\sigma - 2) \\
&\quad + \exp\zeta_E\{(\tfrac{2}{5}\sigma^{-1} - \tfrac{1}{4} - \tfrac{1}{8}\sigma^{-1}\zeta_E)\cos\zeta_E - (\tfrac{9}{40}\sigma^{-1} - \tfrac{1}{4} \\
&\quad + \tfrac{1}{8}\sigma^{-1}\zeta_E)\sin\zeta_E - \tfrac{1}{40}\sigma^{-1}\exp\zeta_E\}, \\
W_{T,1} &= \tfrac{1}{2}(1 - \sigma^{-1})\exp(\tfrac{1}{2}\sqrt{\sigma}\,\zeta_T) + \tfrac{1}{4}(\tfrac{1}{2}\sigma^{-1} - 1), \\
W_1 &= W_{t,1} = \tfrac{1}{8}\sigma^{-1} - \tfrac{1}{4}, \\
W_{e,1} &= \tfrac{1}{8}\sigma^{-1} - \tfrac{1}{4} + \exp(-\zeta_e)\{(\tfrac{1}{4} - \tfrac{3}{20}\sigma^{-1} - \tfrac{1}{8}\sigma^{-1}\zeta_e)\cos\zeta_e + (\tfrac{1}{4} + \tfrac{1}{40}\sigma^{-1} \\
&\quad + \tfrac{1}{8}\sigma^{-1}\zeta_e)\sin\zeta_e + \tfrac{1}{40}\sigma^{-1}\exp(-\zeta_e)\};
\end{aligned}$$

$$\begin{aligned}
F_{E,1} &= \tfrac{1}{2}\sqrt{\sigma}(1 - \zeta_E - \exp\zeta_E\cos\zeta_E), \\
F_{T,1} &= \{(\tfrac{1}{4}\sigma - \tfrac{1}{8})\zeta_T + \tfrac{1}{2}\sqrt{\sigma}\}\exp(\tfrac{1}{2}\sqrt{\sigma}\zeta_T);
\end{aligned}$$

$$\begin{aligned}
H_{E,1} &= -\tfrac{1}{2}h_0\sqrt{\sigma}\,\zeta_E, \\
H_{T,1} &= h_0\exp(\tfrac{1}{2}\sqrt{\sigma}\,\zeta_T)\{\tfrac{1}{2}\sqrt{\sigma}(1 - \sigma^{-1}) - \tfrac{1}{4}\sigma(\tfrac{1}{2}\sigma^{-1} - 1)\zeta_T \\
&\quad - \tfrac{1}{2}\sqrt{\sigma}(1 - \sigma^{-1})\exp(\tfrac{1}{2}\sqrt{\sigma}\zeta_T)\}.
\end{aligned}$$

If we now assume that $\tau < 4$, and that all the l_1, etc., are $\beta^{\frac{1}{2}\tau}$ we see from equations (6.1)–(6.3) that all the first-order terms in V, W and F are zero. Thus there are only two relevant equations

$$0 = H''_{E,1},$$

$$\sigma W_{T,0} H'_{T,1} = 2F_{T,0} + H''_{T,1},$$

with the boundary conditions given by (6.16). Integrating and matching the equations we have

$$H_{E,1} = 0, \quad H_{T,1} = 4\zeta_T\sigma^{-\frac{1}{2}}\exp(\tfrac{1}{2}\sqrt{\sigma}\,\zeta_T).$$

As in § 5.4, since the equations for the first-order variables are linear it follows that, when $\tau = 4$, the solutions are obtained by adding the solutions for $\tau > 4$ and for $\tau < 4$.

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